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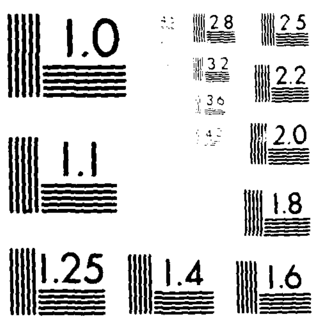

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SATELLITE EPHEMERIS ERRORS
AS THEY RESULT FROM
SURVEILLANCE-SYSTEM MEASUREMENT ERRORS

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THE PREDICTION OF SATELLITE EPHEMERIS ERRORS AS THEY RESULT FROM SURVEILLANCE-SYSTEM MEASUREMENT ERRORS

August 1981

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Approved by
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ABSTRACT

This report derives equations predicting satellite ephemeris error as a function of measurement errors of space-surveillance sensors. These equations lend themselves to rapid computation with modest computer resources. They are applicable over prediction times such that measurement errors, rather than *uncertainties* of atmospheric drag and of Earth shape, dominate in producing ephemeris error. This report describes the specialization of these equations underlying the ANSER computer program, SEEM (Satellite Ephemeris Error Model). The intent is that this report be of utility to users of SEEM for interpretive purposes, and to computer programmers who may need a mathematical point of departure for limited generalization of SEEM.

A

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Dr. Bowen Eugene Simmons
August 1981

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I. INTRODUCTION

A. General Overview

Earth-satellite ephemeris estimation (i.e., position prediction) is fundamental to many space-related operations. Measurements by friendly space-surveillance sensors are computer processed to yield necessary ephemerides. Each ephemeris thus provided has some characteristic accuracy.

The prediction of ephemeris error is also important, both operationally and in the planning of space surveillance systems and of data reduction procedures.

The problem addressed here is the mathematical prediction of ephemeris error, as it results from measurement error alone. The results are valid under conditions where one may validly ignore *uncertainties* of atmospheric drag and of Earth shape. A major requirement was that resulting equations be suitable for computer programming to obtain rapid calculations with modest computer resources.

This report presents a detailed, general parametric solution to the above problem. This report gives, in particular, the somewhat specialized form of that solution, which is the basis of the new ANSER computer program, SEEM (Satellite Ephemeris Error Model).*

SEEM demonstrates, with a time-shared HIS-635 computer, the requisite programming suitability of the mathematical results herein. Reference 1 describes successful use of

* A FORTRAN program, written by the author of this report, as yet unpublished.

SEEM empirically to investigate conditions of its validity for ephemeris error prediction *vis-a-vis* Earth-based radar sensors.*

This report is directed, first, to users of SEEM who may wish to understand its fundamental interpretation. It is also directed to computer programmers who may wish to modify certain specializations of the current version of SEEM and need a mathematical point of departure. The intent is that material here be accessible to engineers and scientists who do not necessarily specialize in either statistics or astrodynamics.

Thus, this report is semi-tutorial in style, and is derivationally self-contained to the extent practical. Some original mathematical developments are included and appropriately noted.

B. Technical Overview

The purpose of this section is to provide not only an overview of report organization, but also a substantive discussion of ephemeris error estimation sufficient (1) for

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* A summary of established SEEM applicability is as follows. The model validly accommodates drag forces for satellite altitudes above about 180 km, over time intervals—encompassing both sensor measurements and the prediction times—of up to at least 9 hours. The model validly accommodates noncentral-forces gravitational fields for low-altitude satellite passes by as many as three Earth-based radars, over somewhat longer measurement-and-prediction time intervals. Assumptions here are (1) current capability for predicting drag forces; (2) current understanding of geoid and other gravitational perturbations; and (3) no radical radar accuracy improvements beyond today's state-of-the-art.

routine interpretation of SEEM inputs and outputs, and (2) for appreciation of options for limited generalization of SEEM.

Three subsections follow. The first gives the overall technical approach. The second overviews report organization, and in particular that of Chapter II. The third discusses substantively Chapter III, the heart of the report.

1. Approach to Solution

The overall approach to solution of the stated problem is to ignore drag and geoid uncertainties by estimating ephemeris errors under the Keplerian (central-force-field) approximation. The rationale for this approach is that the distance of a "Keplerian" satellite from a Keplerian-estimated position should approximate the distance of a "real-world" satellite from a position estimated via perturbation theory, *provided* that calculational treatment of perturbations is exact.

The cited investigations of Reference 1 appear to confirm the validity of the Keplerian approximation for the problem at hand.

The specific analytical approach of this report is parametric, utilizing standard linear-algebra procedures in a covariance-matrix formulation.

2. Organization of the Report

This report constitutes three chapters, plus four appendices. The non-statistician reader may wish to read Appendix A before proceeding to Chapter II. Appendix A is a purely tutorial review of covariance matrix theory.

Chapter II reviews the linear-algebra theory by which one may estimate ephemerides—in a Keplerian universe—from sensor measurements. This chapter serves also to develop the notation used later on. The theory of Chapter II accommodates a variety of sensor types, sensor basing both terrestrial and on satellites, and a variety of "unbiased statistical estimators."

Section II.A defines useful coordinate systems, and some matrix transformations among finite and infinitesimal (error) vectors in those systems. Section II.B, drawing upon Appendix B, deals with transition matrices between astrodynamical state vectors and between error vectors associated with those states.

Section II.C derives the sensor "observation equation." Section II.D treats the iterative differential correction process, which transforms observations into an estimated state vector corresponding to an arbitrary epoch (i.e., instant in time). A subset of the components of this vector constitutes an ephemeris estimate.

Section II.E discusses error minimization criteria and associated unbiased statistical estimators. Finally, Section II.F outlines calculational shortcuts for (1) estimating state vectors for many epochs, and (2) revising state-vector estimates so as to exploit newly available measurement data.

The next subsection describes Chapter III both organizationally and substantively, and also defines the supporting role of Appendix C.

3. Ephemeris Error Analysis

Chapter III deals with prediction of errors in ephemerides that would be arrived at by the methods of Chapter II.* This error analysis assumes perfect convergence of the iterative differential correction process of ephemeris estimation. Thus, the limiting accuracy characteristics of a particular ephemeris estimation process are amenable to assessment, even without analyzing the practical convergence properties of that process.

a. Inputs, Outputs, and General Solution

Section III.A defines in mathematical terms the assumed inputs and desired outputs of the problem.

Key desired outputs are the standard deviations of the components of satellite position error, for the time of each ephemeris. These are to be expressed in a coordinate system selected to make correlations among ephemeris error components vanish. Further desired outputs are the orientation angles of that coordinate system, relative to a "UVW" system defined as having the following axes at an instant in time:

RADIAL - Up: geocenter toward satellite

"ALONG-TRACK" - A third axis orthogonal to the other two axes, approximately along the satellite velocity vector (exact for circular orbits)

* Actual estimation of ephemerides is unnecessary to estimate their errors.

CROSS-TRACK - Normal to the orbital plane, directed along the satellite angular momentum vector.

The above standard deviations and orientation angles have a particularly simple physical interpretation if, in addition to the input assumptions listed below, the probability distributions of the measurement errors are of multivariate normal (Gaussian) form. Then (see Appendix A) one may interpret the standard deviations as the principal half-axial dimensions of a "1σ" ellipsoidal confidence volume, oriented along the axes of the rotated coordinate system. One may interpret the ellipsoid as centered either at the true ephemeris, expressive of a level of confidence that the estimated ephemeris will fall within the ellipsoid; or as centered on the estimated ephemeris, expressive of a level of confidence that the *true* ephemeris will fall within the ellipsoid.

The level of confidence of a 1σ ellipsoid of satellite position is about 20 percent. For many interpretative purposes, a 3σ ellipsoid (i.e., having triple the dimensions of the 1σ ellipsoid) is more useful, providing a confidence level of 61 percent.

Assumed inputs are (1) the "true" orbit parameters of the satellite; (2) sensor types (measuring any subset of the quantities: range, two angles, and their respective rates) and locations (sensors may be fixed or may move on or above the Earth's surface); (3) sensor envelopes of geometric coverage (range and angle extrema); (4) the assumption that sensor measurements are "unbiased"; (5) "true" statistical parameters of measurement error, in the form of a covariance matrix D covering all sensor measurements made; (6) the covariance matrix C—some approximation to D—characterizing the "estimator" via which actual estimation of ephemerides would proceed; and (7) the instants in time corresponding to the presumed ephemerides whose accuracy is in question.

There are still further assumed inputs, which, however, derive calculationally from above inputs (1), (2), and (3). These further inputs are "ideal" (error-free) sensor observation data of the target satellite. A "driver" program to generate these data preexisted SEEM at ANSER.* This report does not give the full mathematical basis of such a driver program, although Sections II.A and II.B do provide many of the necessary transformations.

Regarding inputs (5) and (6) above, the matrices D and C typically contain $>10^6$ elements each. Apparently, their general parametric specification poses a practical difficulty. Actually, in the case of C, this difficulty must in some sense be resolvable relative to any *practical* procedure for ephemeris estimation, since any such procedure involves specification of C (see ensuing discussion of Section III.C).

Section III.B gives the general solution equations for Keplerian ephemeris error prediction. These equations

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* A multipurpose Keplerian program, thus far unpublished. It utilizes a modified Earth rotation rate, thereby correcting to first order for the drift of the satellite orbital plane due to the Earth's equatorial bulge.

provide desired outputs as a function of assumed inputs. These equations appear to present a further practical difficulty, in that they require extraction of the inverse of the large matrix C.*

As before, this difficulty must actually be resolvable for error prediction relative to any practical procedure for ephemeris estimation, since extraction of C^{-1} is necessary there also. However, this difficulty is not necessarily resolvable relative to "ideal" ephemeris estimation, for which one must take C = D.

b. Detailing of the General Solution

Section III.C presents candidate representations of D that resolve the specificational, and in one case also the inversion, difficulties identified above. These representations serve as a point of departure for selection of C representations applicable to ephemeris estimation and, concomitantly, to ephemeris error prediction. The two subsections of Section III.C warrant detailed discussion here.

Subsection III.C.1 begins by assuming that in the "real" Keplerian world, raw measurement data are preprocessed at each sensor for each pass as follows, for entry into the ephemeris estimation process.

Repeatedly, raw data accumulated over an interval of a few seconds are suitably averaged, such that the averaged results correspond to the instant at the center of the observation interval. Known corrections for systematic error

* Practical, not theoretical, invertibility of C is at issue here. Theoretical existence of C^{-1} follows from the definition of C as a covariance matrix, with the stipulation that "perfect" sensors (i.e., having any zero standard deviation of measurement error) are not allowed.

(e.g., sensor calibration corrections) are then applied to the averaged data to form a single "observation vector." If the sensor happens to be a doppler radar, for example, observation components would be range, two angles, and range rate. At the end of the pass, the collection of all observation vectors is then fed into the ephemeris estimation process. Subsection III.C.1 defines D to be the "true" covariance matrix of the errors of all observation vectors, over all passes and sensors.

This subsection then treats the errors of each observation vector as the sum of "noise errors" and "residual bias errors," the latter accounting for all residual systematic errors in the observation. By assumption, noise errors may be correlated with each other within an observation, but not from observation to observation. By further assumption, the noise errors are uncorrelated with residual bias errors, within an observation and from observation to observation. In order to exclude "perfect" sensors, all noise-error standard deviations must be nonvanishing, however small. In order to ensure "unbiased" measurements, it is sufficient to assume that both noise-error and residual-bias error probability distributions are symmetric about zero.

The effect of this decomposition upon D is to render it the sum of a "noise matrix" and a "residual bias matrix." Of these, the noise matrix is block diagonal, each block being the covariance matrix of a single observation and of dimensionality at most 6×6. If the noise errors of an observation happen to be uncorrelated among themselves, the corresponding block will be diagonal.

The residual bias matrix may, however, be relatively complicated, with widespread off-diagonal terms representing long-term correlations among residual-bias errors. Subsection III.C.1 takes a first step toward simplifying this matrix by assuming zero correlation among residual-bias errors of different sensors. Thus, with appropriate organization of D, the residual-bias matrix becomes block diagonal, each block corresponding to all the passes by a particular sensor.

Subsection III.C.1 concludes by developing a detailed parametric representation for the noise and residual-bias matrices, structured as just described. Parameters comprise various error standard deviations and correlation coefficiencies, with general functional dependencies upon satellite position relative to the sensor.

Thus, Subsection III.C.1 provides D structures that are physically realistic for a wide variety of sensing conditions. It also provides a parametric formalism lending itself to practical specification of D as an input to ephemeris error prediction. However, because of the large blocks of elements within the residual bias matrix, the structuring of Subsection III.C.1 is not generally sufficient to provide practical invertibilty of D. Hence, this D-structure does not generally permit "ideal" ephemeris estimation with C = D.

The objective of Subsection III.C.2 is to furtner structure the residual-bias matrix so as to arrive at an easily invertible D, yet accord with physical reality for at least some measurement circumstances.

Subsection III.C.2 assumes, for any given sensor, that the residual bias errors do not change appreciably over a pass, or alternatively over several passes closely spaced in time (a "pass multiplet").* This subsection further assumes that residual biases change significantly between pass multiplets (but their standard deviations do not change) such that residual-bias correlations vanish between multiplets.

Thus, each sensor block of the residual bias matrix decomposes into a set of small blocks, each corresponding to a pass multiplet for that sensor. Each multiplet block constitutes a set of partitions—corresponding to individual observation vectors—that are identical over the entire block.

With the aid of a derivation detailed in Appendix C, Subsection III.C.2 infers and then proves the validity of a closed-form equation yielding D^{-1} as a function (1) of the partitions of the residual bias matrix and (2) of the inverses of the partitions of the noise matrix. (As mentioned earlier, these partitions are of maximum 6×6 dimensionality.) Hence, except for matrix multiplications involving the partitions of the residual-bias matrix, this equation reduces the complexity of extracting D^{-1} (as structured) to that of calculating the inverse of the noise matrix alone.†

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* Note the implication that residual-bias error is insensitive to satellite position relative to the sensor. This assumption may be inappropriate, for example, if it should happen that atmospheric-refraction uncertainties become large at angles near the horizon.

† This equation may be unique to this report. However, a literature search was not feasible within the scope of this analysis effort.

To summarize, Section III.C provides various candidate D-structures, including structures intermediate to those just described, for use in detailed expansion of the general error-prediction equations in Section III.B.

Section III.D continues first by introducing two broad classes of C-structures as approximations to D, and some gradations among them. Section III.D then explicitly details the general solution equations for seven combinations of D-structures and C-structures.

The first class of C-structures constitutes those congruent to the noise matrix of D, i.e., those wnich are block diagonal, elsewhere with zeroes for every element representing correlations from one observation to another. These structures are readily invertible and give rise to what is sometimes referred to as "weighted-least-squares" (WLS) ephemeris estimation. WLS estimation ignores all correlations from one observation to another.

"Simple" WLS estimation, a special case, in addition ignores all correlations among measured quantities within an observation, i.e., it uses a C-structure that is strictly diagonal. This is equivalent to the classical estimation method of Gauss, and produces an optimum fit of the estimated orbit to *actual sensor measurements* (see Section II.E).

The second class of C-structures consists of tnose allowing nonvanishing elements that represent correlations among observations. These structures generally incur practical difficulties of inversion, and their use is not ordinarily attempted in practice. The special case when actually C = D gives rise to "minimum variance" estimation and produces—if calculationally feasible—an optimum fit of the estimated orbit to *the true orbit.*

Because of its ready invertibility, the "pass-multiplet" D-structure of Subsection III.C.2 offers the opportunity for minimum variance estimation when (1) that structure is valid, and (2) parameters of measurement error are known with sufficient accuracy that C becomes D.

Section III.D provides detailed expansions of the general error prediction equation for C-structures that are congruent to each of the D-structures of Section III.C, and in addition for the WLS C-structure, which is congruent to the noise-matrix of them all. All C-structures are distinct from the D-structures, however, in that their parametric values may be different—their parametric sets may even be different for the same congruence constraint.

As the various expansions reveal, the amount of feasible detailing of solutions is quite limited, except for those C-structures whose inverses can be extracted analytically. Those are the WLS C-structure and the "pass multiplet" C-structure. These two expansion cases comprise the point of departure for the specializations of SEEM.

c. <u>The Mathematical Basis of SEEM</u>

Section III.E, the final section of the final chapter of the report, deals with SEEM. Of the two subsections, III.E.2 presents and discusses numeric examples of SEEM outputs. That subsection requires no further discussion here. Subsection III.E.1 gives the analytical specializations of SEEM.

As input, SEEM accommodates only radars, and specifically only those that operate in "altazimuth" coordinates: azimuth, elevation, and range.

One may, however, input a telescope-type sensor by a strategem, i.e., by assigning a very large value to the range measurement error. Thereby, one assigns a low statistical weight to range measurements.

One may also (to some approximation) input other range-and-two-angle coordinates, e.g., angles relative to the boresight of a phased-array radar, by (1) in the driver program, converting to altazimuth coordinates for the "ideal" observation calculations; (2) in the driver program, finding a geometrical coverage volume in altazimuth coordinates that approximates the true coverage volume; and (3) assigning angle errors to their nearest geometric angle analogs in azimuth and elevation.

SEEM allows correlations only among errors of a given measurement component—e.g., range-elevation error correlations are not allowed. Thus, the observation blocks of the D noise matrix are diagonal, as is each small partition of the residual bias matrix.

Via the following additional assumptions, SEEM allows specification of the error performance of each radar in terms of six parameters, the first three being the (constant) residual-bias standard deviations. The remaining three parameters are the noise errors, which are functionally dependent upon satellite position in the radar field of view as follows:

(1) The standard deviation of azimuthal noise error is proportional to 1/cos h, where h is elevation angle (accounting for increasing indeterminacy

of azimuth measurements at elevation angles approaching the zenith). The constant of proportionality is hence the azimuthal standard deviation at 0° elevation.

(2) The standard deviation of elevation noise error is constant, not a function of azimuth, elevation, or range.

(3) The standard deviation of range noise error is constant, not a function of azimuth, elevation, or range.

In the light of (3) above, the present version of SEEM may be inappropriate for high-altitude satellites, where maximum range is set by radar range performance rather than by horizon-limited line-of-sight. (SEEM validation did not include satellite altitudes above approximately 1,000 km.) SEEM defines all pass multiplets as containing just one pass. Thus, the residual bias matrix of D—and hence also D itself—is block diagonal in one-pass blocks.

SEEM provides two choices of C for characterizing the ephemeris estimation process. In the "minimum variance" choice, C = D. In the "least squares" choice, C is set equal to the noise matrix of D. (SEEM validation was conducted only for the least-squares choice.)

SEEM ephemeris error component standard deviations are specified in UVW coordinates only, and do not include correlation coefficients that may not vanish in those coordinates. A further output, the standard deviation of the resultant error vector is invariant with respect to coordinate-system selection. Hence, the UVW resultant error is correct even without coordinate rotation.

Empirical results with SEEM indicate that in fact the error ellipsoid does align itself with the UVW axes soon—in prediction time—after the most recent pass (see Subsection III.E). This alignment is due primarily to the effect of period uncertainty, which makes the along-track error ordinarily large compared to radial and cross-track errors.

II. EPHEMERIS ESTIMATION

This section describes linear-algebra methodology for estimating satellite ephemerides from sensor observations, assuming a Keplerian (central-force field) universe. There is no restriction as to selection of a particular statistical estimator, except that it be "unbiased."

Most of the material here occurs—in one form or another—in References 2 and 3. A first-order correction term to the error transition matrix [i.e., Ψ_{ji} in Equation (26)] does not appear in Reference 2, but is well known in estimation theory. The explicit representation of the partial derivatives of Ψ_{ji} may well be new as developed in Appendix B, but are available elsewhere in somewhat different form [see p. B-6, including footnote].

To the extent practical, the notation here follows the Herrick standards [Reference 3, *Astrodynamical Terminology, Notation and Usage (Appendix)*, pp. 477-511]. The major exception here is that lightface uppercase Roman letters represent various matrices rather than specialized astrodynamical quantities.

A. Reference Frames, Coordinate Systems, and Transformations

Let the time t_0 be the (arbitrary) initial epoch of the analysis. Referring to Figure 1, define a righthanded Cartesian reference frame with positive z-axis through the Earth's North Pole, and positive x-axis intersecting the Greenwich meridian as it happens to lie at t_0.

* This x-axis choice promotes algebraic simplicity. Conversion of ensuing equations to a system with x-axis positive toward the vernal equinox is straightforward.

At time

$$\hat{t} \equiv t - t_o \tag{1}$$

let the satellite position be

$$\mathbf{r} \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

and a sensor position be

$$\mathbf{r}_T \equiv \begin{bmatrix} x_T \\ y_T \\ z_T \end{bmatrix} \tag{3}$$

(One should not consider the sensor position as necessarily on the Earth's surface, although it is so depicted in Figure 1 for ease of geometric interpretation.)

Again referring to Figure 1, define a topocentric (sensor-centered) reference frame as righthanded Cartesian, with positive z'-axis toward the zenith and positive x'-axis toward the South point of the compass. In this frame let the satellite position be

$$\boldsymbol{\rho} \equiv \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad . \tag{4}$$

With these definitions,

$$\mathbf{r} = \mathbf{r}_T + A\boldsymbol{\rho} \quad , \tag{5}$$

FIGURE 1
RELATIONSHIP OF INERTIAL (xyz) TO
TOPOCENTRIC (x'y'z') REFERENCE FRAMES

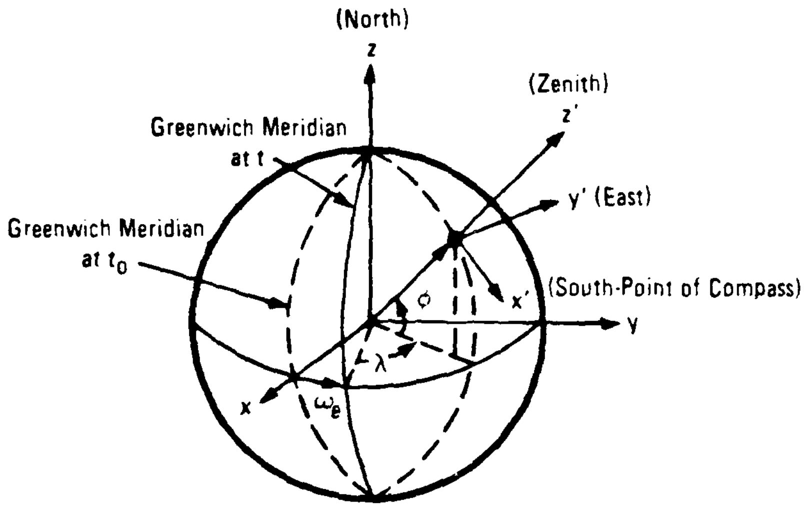
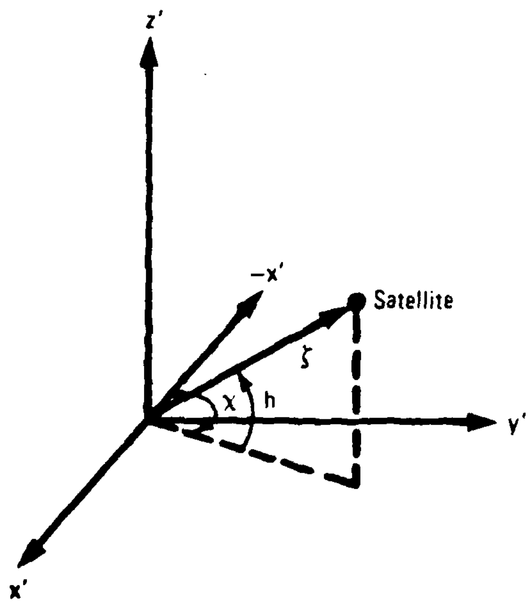


FIGURE 2
ALTAZIMUTH COORDINATES
IN THE TOPOCENTRIC REFERENCE FRAMES



where A is a rotation matrix. Table 1 gives a representation for $\mathbf{r}_T$ and A in terms of sensor longitude λ, latitude ϕ, and geocenter distance r_T.* These may be time-varying quantities.

Suppose now that one regards the components of $\boldsymbol{\rho}$ as functions of an arbitrary set of curvilinear coordinates q_1, q_2, q_3. These will subsequently become the angle and range coordinates characterizing operation of a given sensor. (Their particular significance may, however, differ from sensor to sensor.) Let

$$\mathbf{q} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad . \tag{6}$$

Differentiate Equation (5) with respect to time, obtaining

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_T + \dot{A}\boldsymbol{\rho} + A\, J\dot{\mathbf{q}} \quad , \tag{7}$$

where J is the Jacobian matrix for $\boldsymbol{\rho}$ as a function of $\mathbf{q}$. Table 1 gives representations for $\dot{A}$ and J.

As an illustrative example, consider the case of a sensor fixed at some position on the Earth's surface and designed to operate in altazimuth coordinates. Following the notation of Figure 2, one may write

$$\mathbf{q} = \begin{bmatrix} \chi \\ h \\ \rho \end{bmatrix} \quad , \qquad \dot{\mathbf{q}} = \begin{bmatrix} \dot{\chi} \\ \dot{h} \\ \dot{\rho} \end{bmatrix} \tag{8}$$

---

* The representation of A is a standard result. One may derive it by taking products of elementary rotation matrices, which provide first, a rotation of the primed reference frame about its y'-axis through the angle $-(\pi/2 - \phi)$; and second, a rotation about the (now) z-axis through the angle $-(\lambda + \omega_e\, \hat{t})$.

TABLE 1
TRANSFORMATION QUANTITIES

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$r_T = r_T \begin{bmatrix} \cos\Lambda \cos\phi \\ \sin\Lambda \cos\phi \\ \sin\phi \end{bmatrix}, \Lambda \equiv \lambda + \omega_e \hat{t}$$

$$A = \left[\begin{array}{c|c|c} \cos\Lambda \sin\phi & -\sin\Lambda & \cos\Lambda \cos\phi \\ \sin\Lambda \sin\phi & \cos\Lambda & \sin\Lambda \cos\phi \\ -\cos\phi & 0 & \sin\phi \end{array}\right]$$

$$A^{-1} = A^{\dagger}$$

$$\rho(q) = \begin{bmatrix} x'(q) \\ y'(q) \\ z'(q) \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial\rho}{\partial q_1} & \frac{\partial\rho}{\partial q_2} & \frac{\partial\rho}{\partial q_3} \end{bmatrix}$$

$$\dot{A} = \left[\begin{array}{c|c|c} -\dot{\Lambda} \sin\Lambda \sin\phi + \dot{\phi} \cos\Lambda \cos\phi & -\dot{\Lambda} \cos\Lambda & -\dot{\Lambda} \sin\Lambda \cos\phi - \dot{\phi} \cos\Lambda \sin\phi \\ \dot{\Lambda} \cos\Lambda \sin\phi + \dot{\phi} \sin\Lambda \cos\phi & -\dot{\Lambda} \sin\Lambda & -\dot{\Lambda} \cos\Lambda \cos\phi - \dot{\phi} \sin\Lambda \sin\phi \\ \dot{\phi} \sin\phi & 0 & \dot{\phi} \cos\phi \end{array}\right]$$

$$K = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] , \qquad K_{ij} = \sum_{k=1}^{3} \left(\dot{q}_k \cdot \frac{\partial J_{ik}}{\partial q_j} \right)$$

$$Q = \left[\begin{array}{c|c} AJ & 0 \\ \hline \dot{A}J + A K & AJ \end{array}\right] \qquad Q^{-1} = \left[\begin{array}{c|c} (AJ)^{-1} & 0 \\ \hline (AJ)^{-1}(\dot{A}J + AK)(AJ)^{-1} & (AJ)^{-1} \end{array}\right]$$

Note: ω_e = Earth rotation rate [radians/unit time] and arrive at the specialized transformation matrices of Table 2.

Returning to the general case, introduce the composite vectors

$$\begin{bmatrix} r \\ \hline \dot{r} \end{bmatrix} \quad , \quad \begin{bmatrix} q \\ \hline \dot{q} \end{bmatrix}$$

which will subsequently be useful. Equations (5) and (7) provide a functional relationship between these vectors—unfortunately, a nonlinear relationship since $\rho(q)$ is nonlinear.

However, one can show that a fully linear relationship does exist between the differentials of these vectors, of the form

$$\begin{bmatrix} \delta r \\ \hline \delta \dot{r} \end{bmatrix} = Q \begin{bmatrix} \delta q \\ \hline \delta \dot{q} \end{bmatrix} \quad . \tag{9}$$

These vectors will represent errors at a particular instant, so that in performing differentiations $\hat{t}$ is to be held constant.

To find Q, first differentiate Equation (5), regarding both sensor position r_T and the rotation matrix A as "known" (i.e., error-free) and hence as constants. One obtains

$$\delta r = A J \delta q \quad . \tag{10}$$

Differentiating Equation (7),

$$\delta \dot{r} = \dot{A} J \delta q + A K \delta q + A J \delta \dot{q} \quad , \tag{11}$$

TABLE 2
TRANSFORMATION MATRICES:
EARTH-SURFACE ALTAZIMUTH COORDINATES

$$\boldsymbol{\rho} = \rho \begin{bmatrix} -\cos\chi\cos h \\ \sin\chi\cos h \\ \sin h \end{bmatrix}$$

$$\dot{A} = \omega_e \left[\begin{array}{c|c|c} -\sin\Lambda\sin\phi & -\cos\Lambda & -\sin\Lambda\cos\phi \\ \cos\Lambda\sin\phi & -\sin\Lambda & \cos\Lambda\cos\phi \\ 0 & 0 & 0 \end{array}\right]$$

$$J = \left[\begin{array}{c|c|c} \rho\sin\chi\cos h & \rho\cos\chi\sin h & -\cos\chi\cos h \\ \rho\cos\chi\cos h & -\rho\sin\chi\sin h & \sin\chi\cos h \\ 0 & \rho\cos h & \sin h \end{array}\right]$$

$$J^{-1} = \alpha \left[\begin{array}{c|c|c} \sin\chi & \cos\chi & 0 \\ \cos\chi\sin h\cos h & -\sin\chi\sin h\cos h & \cos^2 h \\ -\rho\cos\chi\cos^2 h & \rho\sin\chi\cos^2 h & \rho\sin h\cos h \end{array}\right]$$

$$\alpha \equiv 1/(\rho\cos h)$$

$$K = \left[\begin{array}{c|c|c} \dot{\chi}\rho\cos\chi\cos h - \dot{h}\rho\sin\chi\sin h + \dot{\rho}\sin\chi\cos h & -\dot{\chi}\rho\sin\chi\sin h + \dot{h}\cos\chi\cos h + \dot{\rho}\cos\chi\sin h & \dot{\chi}\sin\chi\cos h + \dot{h}\cos\chi\sin h \\ \hline -\dot{\chi}\rho\sin\chi\cos h - \dot{h}\rho\cos\chi\cos h + \dot{\rho}\cos\chi\cos h & -\dot{\chi}\rho\cos\chi\sin h - \dot{h}\rho\sin\chi\cos h - \dot{\rho}\sin\chi\sin h & \dot{\chi}\cos\chi\cos h - \dot{h}\sin\chi\sin h \\ \hline 0 & -\dot{h}\rho\sin h + \dot{\rho}\cos\chi & \dot{h}\cos h \end{array}\right]$$

defining a new matrix K via the relation

$$A\,\delta J\dot{q} \equiv A\,K\,\delta q \quad . \tag{12}$$

From Equation (1) one can derive the representation for K given in Table 1.

For the example of the Earth-based altazimuth-type sensor, one can further derive the specialized representation of K given in Table 2.

One can now find the general form of Q by comparing Equations (9), (10), and (11). Table 1 gives the result and also the form of Q^{-1}. One can prove the correctness of Q^{-1} by taking the product QQ^{-1}.

One further transformation will prove useful:

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = L \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad . \tag{13}$$

Here L is the rotation matrix taking the inertial-frame (xyz) representation of $\mathbf{r}$ into a UVW representation (x"y"z") defined as having the unit vector U(x"-axis) directed radially outward from the Earth's center toward the satellite; the unit vector W(z"-axis) directed along the angular momentum vector, normal to the orbital plane; and hence the unit vector V(y"-axis) approximately along-track in the direction $\dot{\mathbf{r}}$—exactly along-track for circular orbits.

To find L in terms of the inertial-frame (xyz) representation of $\mathbf{r}$, begin by defining a quantity

$$\mathbf{s} = \mathbf{r} \times \dot{\mathbf{r}} \quad . \tag{14}$$

(vector cross-product), proportional to the angular-momentum vector. Clearly in the inertial-frame representation,

$$U = \frac{1}{r}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad ; \tag{15}$$

$$W = \frac{1}{s}\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \quad ; \tag{16}$$

$$V = \frac{1}{|s \times r|}\begin{bmatrix} (s \times r)_x \\ (s \times r)_y \\ (s \times r)_z \end{bmatrix} \quad . \tag{17}$$

But the above nine unit-vector components are just the direction cosines among the axes of the two reference frames, and hence are the elements of the rotation matrix relating the frames. That is,

$$L = \begin{bmatrix} U^\dagger \\ \hline V^\dagger \\ \hline W^\dagger \end{bmatrix} \quad , \tag{18}$$

where notationally $U^\dagger$ is the adjoint of U, etc. Hence, L may be evaluated from Equations (12) - (15). Since L is a rotation matrix,

$$L^{-1} = L \quad . \tag{19}$$

B. <u>State Vectors and Related Transformations</u>

One may represent the six parameters of a Keplerian orbit (including the instantaneous position of the satellite in that orbit) by the state vector

$$x_{\hat{t}} = \begin{bmatrix} r \\ \hline \dot{r} \end{bmatrix}_{\hat{t}} \quad , \tag{20}$$

comprising the partitions r, $\dot{r}$. The subscript denotes time functionality. For a satellite in Keplerian orbit, the matrix transformation

$$x_j = \Phi_{ji} \cdot x_i \tag{21}$$

exists. The subscripts denote not matrix elements, but rather the times $\hat{t}_i$ and $\hat{t}_j$. The *propagation matrix* Φ_{ij} has the form

$$\Phi_{ji} = \left[\begin{array}{c|c} fI & gI \\ \hline \dot{f}I & \dot{g}I \end{array}\right]_{ji} , \tag{22}$$

where I is the 3 x 3 identity matrix and Appendix B gives expressions for the scalar functions f, g, $\dot{f}$, and $\dot{g}$.

One can gain some appreciation for Φ_{ji} from the special case of a circular orbit, for which (dropping the explicit I, which one is still to understand as present)

$$(\Phi_{ji})_{\substack{\text{Circular}\\ \text{Orbit}}} = \left[\begin{array}{c|c} \cos(\hat{E}_j-\hat{E}_i) & \frac{1}{n}\sin(\hat{E}_j-\hat{E}_i) \\ \hline -n\sin(\hat{E}_j-\hat{E}_i) & \cos(\hat{E}_j-\hat{E}_i) \end{array}\right] . \tag{23}$$

Here n is the satellite angular rate [radians/unit time], and $\hat{E}_i$, $\hat{E}_j$ are eccentric-anomaly changes since $\hat{t} = 0$.

Important properties of Φ_{ji} are:

<u>Functional Dependence</u>

$$\Phi_{ji} = \Phi(x_i;\hat{E}_j-\hat{E}_i) , \tag{24a}$$

Composition

$$\Phi_{kj}\,\Phi_{ji} = \Phi_{ki} \quad ; \qquad (24b)$$

Inverse

$$\Phi_{ji}^{-1} = \Phi_{ij} \qquad (24c)$$

Determinant

$$|\Phi_{ji}| = 1 \quad . \qquad (24d)$$

Note that when $\hat{E}_j = \hat{E}_i$, Φ_{ji} reduces to the identity matrix.

One can find the transformation between small errors in state vectors by differentiating Equation (19) (i.e., while holding times $\hat{t}_i$ and $\hat{t}_j$ constant):

$$\delta x_j = \Phi_{ji}\delta x_i + (\delta\Phi_{ji})\,x_i \qquad (25)$$

$$\equiv (\Phi_{ji} + \Psi_{ji})\delta x_i \quad . \qquad (26)$$

This defines the important *propagation error matrix* Ψ_{ji}. Appendix B develops an explicit representation for Ψ_{ji} in terms of x_i and $(\hat{E}_j - \hat{E}_i)$.

Table 3 gives a special case of this representation, which in full generality is algebraically lengthy. This case corresponds to a circular equatorial orbit, where the satellite happens to have the position component $x = 0$ [see Equation (2)] at time $\hat{t}_i$ (i.e., the time corresponding to $\hat{E}_i$).

TABLE 3
THE MATRIX Ψ_{ji} (SPECIAL CASE)

$$\Psi = \begin{bmatrix}
3\hat{E}\sin\hat{E} - \sin^2\hat{E} - 2\sin^2\frac{\hat{E}}{2} & 2\sin\hat{E}\sin^2\frac{\hat{E}}{2} & 0 & \frac{2}{n}\sin\hat{E}\sin^2\frac{\hat{E}}{2} & \frac{1}{n}(3\hat{E}\sin\hat{E} - 2\sin^2\hat{E} - 4\sin^2\frac{\hat{E}}{2} & 0 \\
-3\hat{E}\cos\hat{E} + 2\sin\hat{E} + \sin\hat{E}\cos\hat{E} & 4\sin^4\frac{\hat{E}}{2} & 0 & \frac{4}{n}\sin^4\frac{\hat{E}}{2} & \frac{1}{n}(-3\hat{E}\cos\hat{E} + \sin\hat{E} + \sin 2E & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
n(3\hat{E}\cos\hat{E} + 4\sin\hat{E}\cdot\sin^2\frac{\hat{E}}{2}) & n(\cos\hat{E} - \cos^2\hat{E} + \sin^2\hat{E} & 0 & \cos\hat{E} - \cos^2\hat{E} + \sin^2\hat{E} & 3\hat{E}\cos\hat{E} - 3\sin\hat{E} + 8\sin\hat{E}\cdot\sin^2\frac{\hat{E}}{2} & 0 \\
n(3\hat{E}\sin\hat{E} - \cos\hat{E} + \cos^2\hat{E} - \sin^2\hat{E}) & 8n\sin\hat{E}\sin^4\frac{\hat{E}}{2} & 0 & 8\sin\hat{E}\sin^4\frac{\hat{E}}{2} & 3\hat{E}\sin\hat{E} - 2\cos\hat{E} + 2\cos^2\hat{E} - 2\sin^2\hat{E} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

NOTE: $\hat{E} \equiv \hat{E}_j - \hat{E}_i$

Important properties of Ψ_{ji} are:

Functional Dependence

$$\Psi_{ji} = \Psi(x_i;\hat{E}_j-\hat{E}_i) \quad ; \tag{27}$$

Composition

$$(\Phi_{kj} + \Psi_{kj})(\Phi_{ji} + \Psi_{ji}) = \Phi_{ki} + \Psi_{ki} \quad ; \tag{28}$$

Inverse

$$(\Phi_{ji} + \Psi_{ji})^{-1} = \Phi_{ij} + \Psi_{ij} \quad ; \tag{29}$$

Determinant

$$|\Phi_{ji} + \Psi_{ji}| = 1,\ \hat{E}_j = \hat{E}_i \quad ; \tag{30a}$$

$$= [\text{Unbounded as } |\hat{E}_j-\hat{E}_i| \text{ increases without limit}]. \tag{30b}$$

When $\hat{E}_j = \hat{E}_i$, $\Psi_{ji} = 0$.

The inverse follows from the composition property by setting k = i and using Equation (24a).

One may prove the composition property by differentiating Equation (24b) as it operates upon x_i:

$$\delta(\Phi_{ki}x_i) = \delta(\Phi_{kj}\Phi_{ji}x_i) \quad . \tag{31}$$

Now apply the definitional Equation (26) first to the lefthand side, and then repeatedly expand the righthand side:

$$
\begin{aligned}
(\Phi_{ki} + \Psi_{ki})\delta x_i &= \Phi_{kj}[\Phi_{ji}\delta x_i + (\delta\Phi_{ji})x_i] + (\delta\Phi_{kj})\Phi_{ji}x_i \\
&= \Phi_{kj}[(\Phi_{ji} + \Psi_{ji})\delta x_i] + (\delta\Phi_{kj})x_j \\
&= \Phi_{kj}[(\Phi_{ji} + \Psi_{ji})\delta x_i] + \Psi_{kj}\delta x_j \\
&= \Phi_{kj}[(\Phi_{ji} + \Psi_{ji})\delta x_i] + \Psi_{kj}[(\Phi_{ji} + \Psi_{ji})\delta x_i] \\
&= (\Phi_{kj} + \Psi_{kj})(\Phi_{ji} + \Psi_{ji})\delta x_i \quad .
\end{aligned}
\tag{31}
$$

Since δx_i is arbitrary, the compositional Equation (28) must hold.

C. The Observation Equation

Assume that at time $\hat{t}_j$ a sensor measures a subset of the components of the satellite coordinate vector q and its coordinate rate vector $\dot{q}$—for example, azimuth, elevation, range, and range rate. Define ξ as the vector of true values of the measured components and y as the corresponding vector of measurement results. Then

$$
y_j = \xi_j + \eta_j \quad , \tag{32}
$$

where η_j is the measurement error vector and the subscripts note the time of measurement $\hat{t}_j$.

One may now define a matrix M_j, characteristic of the particular sensor making the measurement at $\hat{t}_j$, by the relation

$$\xi_j = M_j \begin{bmatrix} q \\ \dot{q} \end{bmatrix}_j . \tag{33}$$

Then one may write Equation (32) in a more general form, convenient for further development:

$$y_j = M_j \begin{bmatrix} q \\ \dot{q} \end{bmatrix}_j + \eta_j \quad . \tag{34}$$

Table 4 gives examples of M_j for various types of sensors, all of which operate in altazimuth coordinates [see Equation (8)]. That is, the time $\hat{t}_j$ is characterized by <u>sensor type</u>, not only as to the measured component-subset of q_j, $\dot{q}_j$ (specified by M_j), but also by the curvilinear coordinates that q_j represents. Not all sensors operate in altazimuth coordinates, of course. Despite interpretational differences, the *mathematical form* of Equation (34), and *threefold dimensionality of* q_j *therein*, is reasonably general no matter what the value of $\hat{t}_j$.

To proceed, assume next that before the measurement there exists a preliminary orbit determination in the form of a state vector $x_k^{*(1)}$. Here the asterisk denotes an estimated value and the parenthesized superscript denotes an initial estimate. (Methods for preliminary orbit determination are discussed, for example, in Reference 2, Chapters 12 and 13.)

TABLE 4
REPRESENTATIVE (ξ, M) PAIRS

ξ_j	M_j	SENSOR
$\begin{bmatrix} \chi \\ h \end{bmatrix}_j$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	TELESCOPE
$\begin{bmatrix} \chi \\ h \\ \rho \end{bmatrix}_j$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$	RADAR
$\begin{bmatrix} \chi \\ h \\ \rho \\ \dot{\rho} \end{bmatrix}_j$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	DOPPLER RADAR

Now, given $x_k^*(1)$, one may calculate

$$\Phi_{jk}^{*(1)} \equiv \Phi(x_k^{*(1)}; \hat{E}_j - \hat{E}_i) \tag{35}$$

and then, using Equation (21), $x_j^*(1)$. Then one may use Equations (5) and (7) to solve for estimates of q_j and $\dot{q}_j$. Let these estimates be denoted $q_j^*(1)$ and $\dot{q}_j^*(1)$. Finally, form the estimate

$$\xi_j^{*(1)} \equiv M_j \begin{bmatrix} q^{*(1)} \\ \hline \dot{q}^{*(1)} \end{bmatrix}_j \quad . \tag{36}$$

One may then subtract Equation (36) from Equation (34) to obtain

$$\Delta y_j^{(1)} = M_j \begin{bmatrix} q - q^{*(1)} \\ \hline \dot{q} - \dot{q}^{*(1)} \end{bmatrix}_j + \eta_j \quad , \tag{37}$$

where

$$\Delta y_j^{(1)} \equiv y_j - \xi_j^{*(1)} \quad . \tag{38}$$

One can further utilize the known sensor location, together with the estimated quantities of the preceding paragraph, to estimate the matrix $(Q_j^*(1))^{-1}$ (see the formulas of Table 1). In the light of Equation (9), one may write the definition

$$\Delta x_j^{(1)} \equiv \left(Q^{*(1)}\right)^{-1} \begin{bmatrix} q - q^{*(1)} \\ \hline q - q^{*(1)} \end{bmatrix} \tag{39}$$

Note that according to Equation (9),

$$\Delta x_j^{(1)} \equiv x_j - x_j^* \quad , \tag{40}$$

correct to first order. That is, correction terms on the righthand side of higher order in the difference $x_j - x_j^*$ may exist. Using Equation (39), Equation (37) now becomes

$$\Delta y_j = M_j \left(Q^{*(1)}\right)^{-1} \Delta x_j^{(1)} + \eta_j \quad . \tag{41}$$

Suppose one wishes to find, eventually, an estimate of the state vector at an arbitrary time $\hat{t}_i$, an estimate that is to be an improvement over an estimate that is simply

$$x_i^{*(1)} = \Phi_{ik}^{*(1)} x_k^{*(1)} \quad . \tag{42}$$

One then may expect it to be advantageous to introduce $\Delta x_i^{(1)}$ as defined by

$$\Delta x_j^{(1)} \equiv \left(\Phi_{ji}^{*(1)} + \Psi_{ji}^{*(1)}\right) \Delta x_i^{(1)} \quad . \tag{43}$$

This is a first-order version of Equation (26). The parenthesized quantities may be computed from $x_i^{(1)*}$, derived from Equation (42). The quantity Δx_i is of course unknown, from the viewpoint of the satellite observer. Its estimation will be appropriate subsequently.

Using Equation (43), one finally may obtain from Equation (41) the *observation equation*

$$\Delta y_j^{(1)} = T_{ji}^{*(1)} \Delta x_i^{(1)} + \eta_j \quad . \tag{44}$$

Here

$$T_{ji}^{*(1)} \equiv M_j \left(Q_j^{*(1)}\right)^{-1} \left(\Phi_{ji}^{*(1)} + \Psi_{ji}^{*(1)}\right) \quad . \tag{45}$$

In the observation equation, note in summary that one computes $\Delta y_j^{(1)}$ and $T_{ji}^{(1)*}$ from the observation vector y_j and from the initial estimate x_k^* already assumed to be available. The remaining quantities are unknown, although subsequent analysis will assume knowledge of certain statistical properties of η_j.

Further, note that for another observation at, say, time $\hat{t}_m$, one will have redefined the quantity Δx_i such that it may have second-order differences from the Δx_i of Equation (44). The next subsection will ignore such differences in developing an iterative procedure which—if convergent—will eliminate their impact upon an ultimate estimate of x_i.

D. Iterative Differential Correction

Define the following composite quantities for a set of n observations:

$$\Delta y^{(1)} = \begin{bmatrix} \Delta y_1^{(1)} \\ \hline \Delta y_2^{(1)} \\ \hline \vdots \\ \hline \Delta y_n^{(1)} \end{bmatrix} ; \tag{46}$$

$$T_i^{*(1)} = \begin{bmatrix} T_{1i}^{*(1)} \\ \hline T_{2i}^{*(1)} \\ \hline \vdots \\ \hline T_{ni}^{*(1)} \end{bmatrix} ; \tag{47}$$

$$\eta = \begin{bmatrix} \eta_1 \\ \hline \eta_2 \\ \hline \vdots \\ \hline \eta_n \end{bmatrix} . \tag{48}$$

The composite form of Equation (44), j = 1, 2, ⋯, n, is then

$$\Delta y^{(1)} = T_i^{*(1)} \Delta x_i^{(1)} + \eta \quad . \tag{49}$$

Note that this equation places no restrictions upon the time separation of the sensor observations; upon the sensor "mix"; or even that necessarily $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_{n-1} < \hat{t}_n$.

Suppose now that one can find an *estimator* matrix $W_i^{*(1)}$ (i.e., a function of $x_i^*)^{(1)}$, which yields an estimate of $\Delta x_i^{(1)}$ in Equation (49):

$$\left[\Delta x_i^{(1)}\right]^* = W_i^{*(1)} \Delta y^{(1)} \quad ; \tag{50}$$

and which obeys the constraint (not an approximation)

$$W_i^{*(1)} T_i^{*(1)} = 1 \quad . \tag{51}$$

(The righthand side here is an identity matrix.) The next subsection will provide a class of such estimators.

Whatever the specific version of $W_i^{*(1)}$, one can interpret the result of Equation (50) as

$$\left[\Delta x_i^{(1)}\right]^* \cong \left[x_i - x_i^{*(1)}\right]^* \quad , \tag{52}$$

using Equation (40). That is $[\Delta x_i]^*$ is approximately an estimate by which the originally given orbit estimate $x_i^{*(1)}$ [derived from $x_k^{*(1)}$ via Equation (42)] was in error. One might hope that an improved orbit estimate would be

$$x_i^{*(2)} = x_i^{*(1)} + \left[\Delta x_i^{*(1)}\right]^* \quad . \tag{53}$$

One can now repeat the preceding process, replacing $x_i^{*(1)}$ by $x_i^{*(2)}$ and obtaining

$$\left[\Delta x_i^{(2)}\right]^* = W^{*(2)}\Delta y^{(2)} \quad . \tag{54}$$

Continued iteration leads to a sequence

$$x_i^{*(1)}, \; x_i^{*(2)}, \; x_i^{*(3)}, \; \ldots$$

which may converge, i.e.,

$$\lim_{k\to\infty} \left[\Delta x_i^{(k)}\right]^* = 0 \; , \tag{55}$$

depending upon the form of W and upon the accuracy of the initial estimate $x_i^{*(1)}$.

Suppose after, say, k iterations, one stops the iterative sequence, when there remains the estimated error

$$\epsilon^{(k)} \equiv \left[\Delta x_i^{(k)}\right]^* \quad . \tag{56}$$

Then the corresponding form of Equation (50) is

$$\begin{aligned} \epsilon^{(k)} &= W_i^{*(k)}\Delta y \\ &= W_i^{*(k)}\left(T_i^{*(k)}\Delta x_i^{(k)} + \eta\right) \\ &= \Delta x_i^{(k)} + W_i^{*(k)} \cdot \eta \; , \end{aligned} \tag{57}$$

using Equations (49) and (51). The k-iteration analog of Equation (40) is now

$$\Delta x_i^{(k)} \equiv x_i - x_i^* \; , \tag{58}$$

a first-order approximation that may be quite good if substantial convergence has occurred. Substituting into Equation (57) and rearranging,

$$(x_i^{*(k)} - x_i) = W_i^{*(k)} \cdot \eta - \epsilon^{(k)} \quad . \tag{59}$$

This is a result of fundamental importance, since it specifies the error in $x_i^{*(k)}$—i.e., the orbit estimate ultimately obtained from the entire process and that contains within it, as a partition, the ephemeris estimate $r_i^{*(k)}$.

Assume now, and henceforth, that the convergence has been such that $\epsilon^{(k)}$ is small enough to be ignored. Then for simplicity one may denote $x_i^{*(k)}$ as simply x_i^* and $W_i^{*(k)}$ as just W_i^* (i.e., showing its functional dependence upon x_i^*), to obtain

$$(x_i^* - x_i) = W_i^* \cdot \eta \; . \tag{60}$$

Later sections will analyze Equation (60).

E. Estimators

The purpose here is to define and discuss—but not actually to derive—a class of estimator matrices from which one may select a particular member for use in Equation (50). The approach here is first to introduce two specific members of this class, and then to generalize.

Consider a relation of the form

$$w = T\mu + \zeta \; , \tag{61}$$

in which:

- o μ is the true value of an m-vector whose estimate μ^* one wishes to obtain
- o w is a known "measurement" n-vector ($n > m$)
- o ζ is a random "error" n-vector, whose value is unknown but some of whose statistical properties are known
- o T is a known matrix, not a function of μ.

One wants to obtain the estimate μ^* via an estimator matrix W in a relation of the form

$$\mu^* = W \cdot w \ , \tag{62}$$

under some designated optimization criterion.

Moreover, one desires the property that if ζ is unbiased, then the estimation error $(\mu^* - \mu)$ is also unbiased—i.e., one desires that W be an "unbiased estimator."

The unbiased estimator property translates into a simple mathematical constraint. Substituting Equation (61) into Equation (62),

$$\mu^* = WT\mu + W\zeta \ . \tag{63}$$

Then if and only if

$$WT = 1 \ , \tag{64}$$

(the m x m identity matrix),

$$\overline{\mu^* - \mu} = W\bar{\zeta} \ ; \tag{65}$$

and if ζ is unbiased then so is $(\mu^* - \mu)$. Thus, the *exactness constraint* of Equation (64) is a necessary and sufficient condition that W be an *unbiased estimator*.

To define a *weighted least squares* criterion for estimation of μ^*, begin by defining the quantity

$$w^* = T\mu^* \quad , \tag{66}$$

i.e., a noise-free measurement vector corresponding to μ^*. Thus if μ^* is nearly equal to μ, then w^* becomes nearly an ideal measurement. One might reasonably ask that W be chosen such that the magnitude of $w^* - w$ be minimal. One might also ask that those measurement error components corresponding to very accurate measurements be accorded the most statistical weight. That is, one might require that W minimize

$$\sum_{i=1}^{n} \frac{(w_i^* - w_i)^2}{(\sigma_\zeta)_i^2} \quad ,$$

where $(\sigma_\zeta)_i^2$ is the known variance of the i^{th} measurement.

By carrying out an appropriate minimization procedure while observing the exactness constraint (see Reference 2, pp. 201-203), one can obtain the result

$$\begin{aligned} W_{LS} &\equiv (W)_{\text{Weighted Least Squares}} \\ &= (T^\dagger C_{LS}^{-1} T)^{-1} T^\dagger C_{LS}^{-1} \quad . \end{aligned} \tag{67}$$

Here by definition C_{LS} is the n x n diagonal matrix with nonvanishing elements

$$(C_{LS})_{ii} = (\sigma_\zeta)_i^2 \quad , \tag{68}$$

Several features of this result are significant. First, it clearly satisfies the exactness constraint. Second, the statistical properties of ζ that must be known are its component variances (i.e., knowledge of the form of its probability distribution is not necessary). Third, the form of W_{LS} makes the estimation result invariant with regard to selection of w-component dimensional scale (e.g., km or NM). (Note that if the weights $1/(\sigma_\zeta)_i^2$ are arbitrarily set equal to unity as in "ordinary least squares" estimation, dimension-scale invariance no longer holds.) Fourth, the inverse of C_{LS} is trivial to find, so that numerical evaluation of W_{LS} is straightforward even when the number of measurements is large.

One may, however, adopt a different estimation criterion, and arrive at a somewhat different result. Suppose one decides to minimize, not the measurement residuals, but the state-vector residuals—i.e., the individual component variances of the estimation error $(\mu^* - \mu)$. For *minimum-variance* estimation, one is to minimize each of the quantities

$$(\mu_i^* - \mu_i)^2 , \quad i=1, 2, \ldots n ,$$

again observing the exactness constraint.

If one carries out an appropriate minimization procedure (see Reference 2, pp. 185-192) assuming now that the "true" ζ covariance matrix D is known, one can obtain the result

$$W_{MV} \equiv (W)_{\text{Minimum Variance}} \tag{69}$$

$$= (T^\dagger D^{-1} T)^{-1} \, T^\dagger D^{-1} \quad .$$

Significant features of this estimator are as follows. First, it clearly satisfies the exactness constraint. Second, the statistical properties of ζ that must be known are its entire covariance matrix (not, as for W_{LS}, merely the diagonal elements of D). Third, W_{MV} provides a result that is properly invariant with respect to dimensional scale changes. Fourth, the inverse of D may not be trivial to find when the number of measurements is large, so that numerical evaluation of W_{MV} may not be straightforward. Fifth, *the variance of each component of* $(\mu^* - \mu)$ *is indeed minimum for* W_{MV}*, as compared to the* $(\mu^* - \mu)$*-component variances of any other estimator* (including W_{LS}). However, how can one obtain D with assurance?

In fact, D will not be exactly known in practice, but may be approximated by some matrix C that must be real, symmetric, invertible, and have positive diagonal elements. Then the practical estimator will be

$$W = \left(T^{\dagger}C^{-1}T\right)^{-1} T^{\dagger}C^{-1} \quad . \tag{70}$$

This reduces to W_{LS} if $C = C_{LS}$, and W_{MV} if $C = D$, but in fact represents a *class of estimators* where C is selectable. Ease of calculation and estimation accuracy both depend upon selection of an appropriate C.

One may show (see Reference 2, p. 202) that the estimator of Equation (70) results from minimization of the quadratic form

$$(w - T\mu)^{\dagger}C^{-1}(w - T\mu) \quad , \tag{71}$$

subject to the exactness constraint. This generalized optimization criterion is known, somewhat confusingly, as the *weighted least squares criterion*.

With regard to the preceding subsection, the following correspondences hold for the k^{th} iteration:

$$[\Delta x_i^{(k)}]^* \leftrightarrow \mu^* \quad ; \tag{72}$$

$$\Delta y \leftrightarrow w \quad ; \tag{73}$$

$$\eta \leftrightarrow \zeta \quad ; \tag{74}$$

$$T_i^{*(k)} \leftrightarrow T \quad ; \tag{75}$$

$$W_i^{*(k)} \leftrightarrow W \quad ; \tag{76}$$

Note that since a value for $x_i^{*(k)}$ is assumed as an input to each iteration, $T_i^{*(k)} \equiv T(x_i^{*(k)})$ is a known quantity as assumed in the minimization of the quadratic error expression of Equation (71).

The preceding discussion has not addressed three key questions. Does convergence occur, in the iterative differential correction process, for an arbitrary selection of C? If, for a given C, convergence does occur, does it necessarily yield a unique x_i^*? (That is, does Equation (60) have more than one solution?) If x_i^* is unique for a given C, what is the minimization criterion to which x_i^* corresponds?

In fact, convergence may or may not occur for a given selection of C. Further discussion of this topic is beyond the scope of this paper.

One may show, however, (see Reference 2, pp. 437-440), that if convergence does occur, the limit x_i^* is unique for that particular C. Moreover, the resulting x_i^* minimizes the quadratic error expression of Equation (71), wherein T is now to be interpreted as $T(x_i^*)$.

Note, in closing, that all of the analysis of this subsection presupposes that the correct functional form of T is known.

F. Calculational Strategies for Ephemeris Estimation

Suppose one must obtain ephemeris predictions for a sequence of times $\hat{t}_k$, k = 1, 2, ...—that is, suppose one must obtain a number of estimates x_k^* from a given set of measurements. Suppose, moreover, that during the time interval of prediction, additional measurement data occasionally become available. One then desires to obtain a revised set of predictions x_k^* in near-real-time with the arrival of new data.

Calculational efficiency of ephemerides now becomes an issue: is it necessary repeatedly to carry out the full iterative differential correction process for each x_k^*?

Calculational strategies do exist to alleviate this problem, at least under some circumstances. The first strategy simplifies calculation of a set of x_k^* from a given set of measurement data. The procedure is first to find one state-vector, say x_i^*, via iterative differential correction, and then repeatedly to utilize the relation

$$x_k^* = \Phi_{ki}^* \, x_i^*, \quad k=1,2,\ldots, \tag{77}$$

[see Equation (42)].

One would hope to obtain in this manner—independent of the choice of i—the same set x_k^* as by direct use of iterative differential correction for each x_k^*. A proof of this equivalence will conclude this subsection. The strategy of Equation (77) is of quite general utility, involving no restrictions as to the nature of C employed in W [see Equation (70)].

Two further strategies, the "Bayesian filter" and the "Kalman filter without driving noise," do involve such restrictions. The purpose of those strategies is to minimize the recalculation of x_i^* for Equation (76) when, from time to time, new measurement data become available. The key to their utility is treatment of each batch of new data as having errors uncorrelated with errors of all previous data.

Thus, these strategies are useful for particular, block-diagonal forms of C. For such forms, these strategies provide estimation results more expeditiously than, but identical with, complete "brute force" re-estimation of x_i^* from old and new raw data for a given C. Reference 2, Chapters 10 through 12, contains a detailed discussion of the Bayesian filter and the Kalman filter without driving noise. Their further discussion is not appropriate here.

The promised equivalence proof will demonstrate that x_k^* is identical, whether arrived at by iterative differential correction or indirectly via Equation (77). That is, suppose that iterative differential correction yields directly at $\hat{t}_k$ a value x_k^*, and directly at $\hat{t}_i$ the value x_i^*, whence a value x_k^* obtains via Equation (76). The problem is to prove that

$$x_k^{*'} = x_k^* \quad . \tag{78}$$

Now from Equation (60),

$$x_i^* - x_i = W_i^* \cdot \eta \quad , \tag{79}$$

and

$$x_k^{*\prime} - x_k = W_k^{*\prime} \cdot \eta \quad , \tag{80}$$

where the W-arguments are respectively x_i^* and $x_k^{*\prime}$. Given Equation (77), then by Equation (26) to first order

$$x_k^* - x_k = \Xi_{ki}^* \cdot (x_i^* - x_i) \quad , \tag{81}$$

where

$$\Xi_{ki}^* \equiv \Phi_{ki}^* + \Psi_{ki}^* \quad . \tag{82}$$

Premultiplying Equation (79) by Ξ_{ki}^* and combining with Equation (81),

$$x_k^* - x_k = \Xi_{ki}^* W_i^* \cdot \eta \quad . \tag{83}$$

Now from Equation (70) and the correspondences of Equations (72) to (76), the general form of W_i^* is

$$W_i^* = \left(T_i^{*\dagger} C^{-1} T_i^*\right)^{-1} T_i^{*\dagger} C^{-1} \quad . \tag{84}$$

Using this, one may expand the lefthand side of Equation (84) as follows:

$$\Xi^*_{ki}\, W^*_i = \Xi^*_{ki}\left(T^{*\dagger}_i C^{-1} T^*_i\right)^{-1} T^{*\dagger}_i C^{-1}$$

$$= \left(T^{*\dagger}_i C^{-1} T^*_i \cdot \Xi^{*-1}_{ki}\right)^{-1} \left[\left(\Xi^{*\dagger}_{ki}\right)\cdot\left(\Xi^{*\dagger}_{ki}\right)^{-1}\right] T^{*\dagger}_i C^{-1}$$

$$= \left[\left(T^*_i \cdot \Xi^*_{ki}\right)^{\dagger} C^{-1}\left(T^*_i \cdot \Xi^*_{ik}\right)\right] \left(T^*_i \cdot \Xi^*_{ik}\right)^{\dagger} C^{-1} \quad . \tag{85}$$

The last step involves substitution of two subsidiary relations. The first is

$$\Xi^{*-1}_{ki} = \Xi^*_{ik} \quad , \tag{87}$$

a restatement of Equation (29). The second is*

$$\left(\Xi^{*\dagger}_{ki}\right)^{-1} = \left(\Xi^{*-1}_{ki}\right)^{\dagger} \quad . \tag{87}$$

Now from Equations (45) and (47),

$$T^*_i \cdot \Xi^*_{ik} = T^*_k \quad . \tag{88}$$

Substituting this into Equation (85), one finally has

$$\begin{aligned} \Xi^*_{ki}\, W^*_i &= \left(T^{*\dagger}_k C^{-1} T^*_k\right)^{-1} T^{*\dagger}_k C^{-1} \\ &= W^*_k \quad , \end{aligned} \tag{89}$$

---

* This relation holds for any invertible matrix A. Taking the transpose of both sides of

$$A\, A^{-1} = 1 \quad ,$$

one has

$$(A^{-1})^{\dagger} A^{\dagger} = 1 \quad .$$

Hence $(A^{-1})^{\dagger}$ is the inverse of $A^{\dagger}$, as in

$$(A^{-1})^{\dagger} = (A^{\dagger})^{-1} \quad .$$

using the definitional Equation (84). Upon substitution into Equation (83),

$$x_k^* - x_k = W_k^* \cdot \eta \quad , \tag{90}$$

a relation identical in form with Equation (80). But as stated earlier, such an equation has a unique solution, and Equation (78) therefore must hold. This completes the proof as required.

III. EPHEMERIS ERROR PREDICTION

The preceding chapter describes linear-algebra methodology for estimating Keplerian satellite ephemerides from sensor measurements. Specific estimation techniques within this methodology depend upon selection of a matrix C, which is some approximation to D, the covariance matrix of measurement errors [see Equation (70) ff.].

The purpose here is to develop equations for estimating errors in the ephemerides that would be arrived at by the methodology of Chapter II.

Of the five ensuing sections, Section III.A defines the error estimation problem in terms of inputs and outputs. Section III.B derives equations of the general solution, expressed in terms of the covariance matrices C and D. Section III.D introduces specific representations for C and D. Section III.E details the equations of the general solution in terms of those representations. Finally, Section III.F defines and discusses the ephemeris error equations underlying the computer model SEEM.

The general solution of Section III.B here is well known [see Equation (16), Reference 4]. A contribution of this analysis is the representational discussion of Section III.C, and specifically the analytical matrix-inverse given by Equations (143) and (144). When used in the estimator W, it affords calculational efficiency plus some accuracy improvement over the conventional least-squares approach to emphemeris estimation.

The ensuing analysis assumes reader familiarity with covariance matrix theory as reviewed in Appendix A.

A. The Error Estimation Problem

This section defines the ephemeris error analysis problem in terms of assumed inputs and required outputs.

1. Assumed Inputs

Assume that the following inputs are available:

a. The epoch set $\hat{t}_k$, $k = 1,2,\ldots,$ for which ephemeris errors are desired

b. A state vector x_o corresponding to some epoch $\hat{t}_o$ (affording a complete specification of the "true" orbit of the satellite)

c. For each $\hat{t}_j$ at which a measurement is made, the set of quantities [see Sections II.A and II.C]:

$$q_j,\ \dot{q}_j,\quad M_j,\ \Lambda_j,\ \phi_j,\ r_{Tj}$$

(affording a description of "true" observables, the subset of these actually observed, and the sensor position)

d. For each sensor the functional forms of Table 1 necessary to evaluate the matrix Q_j^{-1} from q_j, $\dot{q}_j$ (affording subsequent evaluation of $T_{ji}(x_j)$ [see Equation (45)]

e. Relative to the measurement error vector η: its "true" covariance matrix D; its covariance matrix representation C used in the estimator W; the assumption that η is unbiased, i.e., that $\overline{\eta} = 0$.

The prior generation of input c. from appropriate sensor characteristics is a standard space-surveillance problem not within the scope of this report (although the equations of Sections II.A and II.B are useful in solving that problem).

The assumption of e. that η is unbiased is subject to the following interpretation. Suppose each observation error-vector η_j [a partition of η_j: see Equation (34)] is what remains after application of calibration, atmospheric refraction, and other *known bias* corrections to the raw observation data. The error η_j then comprises "noise" and "residual bias" contributions (see Section III.C). If the probability distribution of each of these is symmetric about zero, then $\overline{\eta}_j = 0$ for each $\hat{t}_j$ and hence $\overline{\eta} = 0$.

2. <u>Desired Outputs</u>

Let S_k denote the covariance matrix of the error $(x_k^* - x_k)$. Let ${}^rS_k$ be the upper lefthand 3 x 3 partition of S_k; i.e., let the elements

$$\left({}^rS_k\right)_{mn} \equiv \left(S_k\right)_{mn}; \; m, n = 1, 2, 3 \quad . \tag{91}$$

Then ${}^rS_k$ is the covariance matrix of ephemeris error, since

$$\begin{aligned} {}^rS_k &\equiv E\left\{(x_k^* - x_k)(x_k^* - x_k)^\dagger\right\} \\ &= E\left\{\left[\begin{array}{c} r_k^* - r_k \\ \hline \dot{r}_k^* - \dot{r}_k \end{array}\right]\left[\begin{array}{c} r_k^* - r_k \\ \hline \dot{r}_k^* - \dot{r}_k \end{array}\right]^\dagger\right\} \\ &= E\left\{\left[\begin{array}{c|c} (r_k^* - r_k)(r_k^* - r_k)^\dagger & (r_k^* - r_k)(\dot{r}_k^* - \dot{r}_k)^\dagger \\ \hline (\dot{r}_k^* - \dot{r}_k)(r_k^* - r_k)^\dagger & (\dot{r}_k^* - \dot{r}_k)(\dot{r}_k^* - \dot{r}_k)^\dagger \end{array}\right]\right\} . \end{aligned} \tag{92}$$

Desired simulation outputs are

a. The matrices ${}^{r}S_k$, corresponding to the desired epochs $\hat{t}_k$, $k = 1,2,...$

b. "Error ellipsoid" interpretation parameters (orientation angles, semi-major axes) for each ${}^{r}S_k$, where orientation is specified relative to the UVW reference frame [see section I.A].

Note (see Appendix A) that the interpretation of ${}^{r}S_k$ in terms of an error ellipsoid centered at r_k is legitimate only when the probability distribution associated with $(r_k^* - r_k)$ is normal and unbiased. This condition is met whenever the probability distribution associated with η is normal and unbiased, since by Equation (60) the error $(x_k^* - x_k)$ [containing $(r_k^* - r_k)$ as a partition] is a linear transformation upon η.

B. <u>General Solution</u>

The purpose here is to derive general equations giving desired simulation outputs as a function of assumed simulation inputs.

Consider Equation (60), written for an arbitrary epoch $\hat{t}_i$:

$$x_i^* - x_i = W_i^* \eta \ . \tag{93}$$

Subsection II.F has established the formal invariance of this first-order approximation [see Equation (58)] under the epoch transformation Equation (26), itself induced by Equation (21).

The theorem represented by Equation (A-18) allows one immediately to write

$$S_i = W_i^* D W_i^{*\dagger} . \tag{94}$$

The lefthand side is a desired simulation output, but the righthand side depends on the quantity x_i^*—according to Section III.A, not an available input.

But via the Taylor expansion in vector form,

$$W_i^* \equiv W(x_i^*)$$

$$\cong W(x_i) + \left[\frac{\partial W}{\partial x_i^*}\right]_{x_i} (x_i^* - x_i) \ . \tag{95}$$

One can see that the order of approximation of Equation (93) is preserved in writing

$$x_i^* - x_i = W_i \, \eta \ . \tag{96}$$

The W_i here is a function of x_i, not x_i^*:

$$W_i \equiv W(x_i)$$

$$= \left(T_i^\dagger C^{-1} T_i\right)^{-1} T_i^\dagger C^{-1} \tag{97}$$

(see Section II.E), with T_i a column matrix whose general partition is

$$\left(T_i\right)_j = M_j \, Q_j^{-1} \, \Xi_{ji} \tag{98a}$$

[see Equations (45) and (82)], where

$$\Xi_{ji} = \Phi_{ji} + \Psi_{ji} \ . \tag{98b}$$

Evaluation of Equations (98) is for

$$x_j = \Phi_{jo} \, x_o \ . \tag{99}$$

Thus

$$S_i = W_i \, D \, W_i^{\dagger} \quad ; \tag{100}$$

and since

$$x_k^* - x_k = \Xi_{ki}(x_i^* - x_i) \quad , \tag{101}$$

one has

$$S_k = \Xi_{ki} \, S_i \, \Xi_{ki}^{\dagger} \tag{102}$$

Equations (100) and (102), together with Equations (91), (97), and (98), give the desired outputs a. as a function of the assumed inputs a., b., c., d., and e.

It remains in this subsection to obtain the output b. from ${}^{r}S_k$ expressed thus far relative to the inertial (xyz) frame.

By Equations (13) and (A-18),

$$\left[{}^{r}S_k\right]_{UVW} = L \, {}^{r}S_k \, L^{\dagger} \quad . \tag{103}$$

Diagonalization of $\left[{}^{r}S_k\right]_{UVW}$, if carried out by an appropriate numerical procedure, yields eigenvalues $(\sigma_1)^2$, $(\sigma_2)^2$, $(\sigma_3)^2$ and corresponding normalized eigenvectors, which one may denote as e_1, e_2, e_3. According to the analysis of Appendix A, the semi-major axes of the "1-σ" ellipsoid have values σ_1, σ_2, σ_3.

One way to interpret the ellipsoid orientation is as follows. Pick the first eigenvector e_1, and denote the angles between e_1, and **U**, **V**, **W** as θ_{11}, θ_{12}, and θ_{13}, respectively. Then

$$\cos\theta_{11} = e_1^{\dagger} \mathbf{U} , \tag{104a}$$

$$\cos\theta_{12} = e_1^{\dagger} \mathbf{V} , \tag{104b}$$

$$\cos\theta_{13} = e_1^{\dagger} \mathbf{W} ; \tag{104c}$$

where

$$\mathbf{U} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad \mathbf{V} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \quad \mathbf{W} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{105}$$

These are the direction cosines of the orientation of the σ_1-axis of the error ellipsoid. Note that one may arbitrarily change the sign of e_1 if ease of interpretation of the angles is improved thereby. Such a change does not upset the normalization of e_1 and physically means that one may take either of the "σ_1-ends" of the ellipsoid to be "positive."

One can interpret each of the remaining eigenvectors similarly, completing extraction of desired outputs b.

C. Measurement Covariance Matrix Structures

The purpose here is to introduce some candidate algebraic structures for the "true" covariance matrix D. These, perhaps with still further approximations, are also candidate structures for C.

The first of the following subsections introduces structures by consideration of the "noise" and "residual bias" concepts mentioned earlier. The second subsection further details this structure into a form which, although of limited generality, does permit analytical extraction of the matrix inverse—of importance since one would like to use W with C = D.

1. Basic Structure

Let each observation epoch $\hat{t}_j$ now be denoted as ${}^{\alpha}\hat{t}_j^m$, where α indexes the sensor making the measurement; m indexes satellite "pass" through the field of view of that sensor; and j is now to be regarded as indexing an observation within a pass.

Consider the structure of D first with regard to partitions corresponding to each observation epoch ${}^{\alpha}\hat{t}_j^m$, and then with regard to the "microstructure" within such partitions.

a. The General Observation Partition

Let the observation error vector ${}^{\alpha}\eta_j^m$ correspond to the epoch ${}^{\alpha}\hat{t}_j^m$. Let the ordering of the ${}^{\alpha}\eta_j^m$ within the composite vector η [see Equation (48)] be hierarchic, such that α varies the slowest, m the next slowest, and j the fastest. The ordering of other composite vectors and of the matrix D will of course correspond. Note that freedom to select this indexing hierarchy exists, since up to this point the analysis has not restricted interpretation of the sequence $\hat{t}_j$ (see II.D).

Now introduce the decomposition

$$ {}^{\alpha}\eta_j^m = {}^{\alpha}\nu_j^m + {}^{\alpha}\epsilon_j^m \quad , \qquad (106) $$

wi · the requirement that the term ${}^{\alpha}\epsilon_j^m$ account for any correlations that may exist from one observation to another. That is, ${}^{\alpha}\nu_j^m$ is to be interpreted as a "noise" term and ${}^{\alpha}\epsilon_j^m$ is to be the "residual bias" term within ${}^{\alpha}\eta_j^m$. It follows that

$$E\left\{\left({}^{\alpha}\nu_j^m\right)\left({}^{\beta}\nu_k^n\right)^{\dagger}\right\} = {}^{\alpha}G_j^m\,\delta_{\alpha\beta}\,\delta_{mn}\,\delta_{jk} \quad , \tag{107}$$

where

$${}^{\alpha}G_j^m \equiv E\left\{\left({}^{\alpha}\nu_j^m\right)\left({}^{\alpha}\nu_j^m\right)^{\dagger}\right\} \quad . \tag{108}$$

Mathematically, this decomposition entails to this point no loss of generality. Physically, the desirable interpretation is that the ${}^{\alpha}\nu_j^m$ result from random receiver and possible random external noise sources, with each "observation" actually deriving from some small data set such that, for appropriate observation spacing, Equation (107) holds. This physical interpretation clearly implies some practical constraints as to signal processing, and moreover implies that always ${}^{\alpha}G_j^m$ is positive definite.

Regarded as residual bias errors, the ${}^{\alpha}\epsilon_j^m$ by contrast will be correlated with each other from one observation to another, at least for observations not too widely separated in time and made by the same sensor. Thus, one might set

$$E\left\{\left({}^{\alpha}\epsilon_j^m\right)\left({}^{\beta}\epsilon_k^n\right)^{\dagger}\right\} = \delta_{\alpha\beta}\,E\left\{\left({}^{\alpha}\epsilon_j^m\right)\left({}^{\alpha}\epsilon_k^n\right)^{\dagger}\right\} \tag{109}$$

$$\equiv \delta_{\alpha\beta}\,{}^{\alpha}H_{jk}^{mn} \quad , \tag{110}$$

assuming no correlations from one sensor to another.

Then by the symmetry of Equation (110),

$$ {}^{\alpha}H_{jk}^{mn} = {}^{\alpha}H_{kj}^{nm} \quad . \tag{111} $$

Also, ${}^{\alpha}H_{jj}^{mm}$ will be non-negative definite, taking into account that residual bias errors may sometimes vanish.

Finally, assume that the noise errors are uncorrelated with the residual bias errors:

$$ E\left\{\left({}^{\alpha}\nu_{j}^{m}\right)\left({}^{\beta}\epsilon_{k}^{n}\right)^{\dagger}\right\} = 0 \quad . \tag{112} $$

Taking all of these relations into account, the general observational partition of D is then

$$ {}^{\alpha\beta}D_{jk}^{mn} \equiv E\left\{\left({}^{\alpha}\eta_{j}^{m}\right)\left({}^{\beta}\eta_{k}^{n}\right)^{\dagger}\right\} \tag{113} $$

$$ = \delta_{\alpha\beta}\left({}^{\alpha}G_{j}^{m}\,\delta_{mn}\,\delta_{jk} + {}^{\alpha}H_{jk}^{mn}\right) \quad , \tag{114} $$

with

$$ {}^{\alpha\beta}D_{jk}^{mn} = {}^{\beta\alpha}D_{kj}^{nm} \quad . \tag{115} $$

The resulting D is—as required—real, symmetric, and invertible, and has positive terms on the diagonal.

Note that sufficient conditions for η to be unbiased are

$$ \overline{{}^{\alpha}\nu_{j}^{m}} = 0 \tag{116} $$

and

$$\overline{{}^{\alpha}\epsilon_j^m} = 0 \quad . \tag{117}$$

Physically, one can regard these as averages for each sensor over large ensembles of observations, with probability distributions symmetric about zero.

b. Microstructure

Now consider the problem of representing structure within the matrices ${}^{\alpha}G_j^m$ and ${}^{\alpha}H_{jk}^{mn}$. As will soon appear, there is a problem in establishing a reasonable notation in which that structure may be specified. This problem will receive primary attention here. Possible functional dependencies of certain quantities will receive limited consideration.

In the present notation, observables at ${}^{\alpha}\hat{t}_j^m$ are ${}^{\alpha}q_j^m$ and ${}^{\alpha}\dot{q}_j^m$, representing a range and two angle variables, and the rates thereof. Some subset (possibly all) of these six quantities is actually measured, resulting in the errors of Equation (106). If one indexes the components of that Equation by p, p = 1,2,..., $\leq$ 6, then*

$$\left({}^{\alpha}\eta_j^m\right)_p = \left({}^{\alpha}\nu_j^m\right)_p + \left({}^{\alpha}\epsilon_j^m\right)_p \quad . \tag{118}$$

One may now introduce a useful representation for ${}^{\alpha}G_j^m$ as follows.

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* **The index p relates to the components of ${}^{\alpha}q_j^m$ and ${}^{\alpha}\dot{q}_j^m$ not directly, but via the sensor characterization matrix ${}^{\alpha}M_j^m$, previously denoted M_j [see Equation (33)].**

Define a matrix of standard noise deviations ${}^{\alpha}_{\nu}\Sigma^{m}_{j}$ as having the general element

$$\left({}^{\alpha}_{\nu}\Sigma^{m}_{j}\right)_{pq} = \delta_{pq}\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)_{p} \quad , \tag{119}$$

where

$$\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)_{p} \equiv \left[E\left\{\left({}^{\alpha}\nu^{m}_{j}\right)_{p}\left({}^{\alpha}\nu^{m}_{j}\right)_{p}\right\}\right]^{1/2} \quad . \tag{120}$$

(Here q = 1,2,..., an index having nothing to do with the vector q.)

Define a matrix of noise correlation coefficients ${}^{\alpha}_{\nu}R^{m}_{j}$ as having the general element

$$\left({}^{\alpha}_{\nu}R^{m}_{j}\right)_{pq} = 1, \; p = q \quad ; \tag{121a}$$

otherwise

$$\left({}^{\alpha}_{\nu}R^{m}_{j}\right)_{pq} = \frac{E\left\{\left({}^{\alpha}\nu^{m}_{j}\right)_{p}\left({}^{\alpha}\nu^{m}_{j}\right)_{q}\right\}}{\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)_{p}\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)_{q}} \quad . \tag{121b}$$

Then from Equation (108),

$${}^{\alpha}G^{m}_{j} = {}^{\alpha}_{\nu}\Sigma^{m}_{j} \cdot {}^{\alpha}_{\nu}R^{m}_{j} \cdot {}^{\alpha}_{\nu}\Sigma^{m}_{j} \quad . \tag{122}$$

This representation of ${}^{\alpha}G^{m}_{j}$ separates the standard deviations of the components of ${}^{\alpha}\nu^{m}_{j}$ from the correlations among them. One may expect that the correlation matrix will depend upon the control-system design of the specific sensor α. If the components of ${}^{\alpha}\nu_j^m$ are uncorrelated, ${}^{\alpha}_{\nu}R_j^m$ reduces to the identity matrix. One may further expect that the size of the standard deviations of error components will depend in general upon signal strength and receiver noise. That is, one expects that ${}^{\alpha}_{\nu}\Sigma_j^m$ will be a sensor-dependent function of ${}^{\alpha}q_j^m$, ${}^{\alpha}\dot{q}_j^m$, and perhaps also of some parameter set ${}^{\alpha}\Omega$ associated with the target—e.g., radar cross-section.

A representation for ${}^{\alpha}H_{jk}^{mn}$, analogous to Equation (122), is obtained as follows.

Define a matrix of residual-bias standard deviations ${}^{\alpha}_{\varepsilon}\Sigma_j^m$ as having the general element

$$\left({}^{\alpha}_{\varepsilon}\Sigma_j^m\right)_{pq} = \delta_{pq}\left({}^{\alpha}_{\varepsilon}\sigma_j^m\right)_p \quad , \tag{123}$$

where

$$\left({}^{\alpha}_{\varepsilon}\sigma_j^m\right)_p \equiv \left[E\left\{\left({}^{\alpha}\varepsilon_j^m\right)_p\left({}^{\alpha}\varepsilon_j^m\right)_p\right\}\right]^{1/2} \quad . \tag{124}$$

Define a matrix of residual-bias correlation coefficients ${}^{\alpha}_{\varepsilon}R_{jk}^{mn}$ as having the general element

$$\left({}^{\alpha}_{\varepsilon}R_{jh}^{mn}\right)_{pq} = 1 \quad \text{if } p = q \text{ and } m = n \text{ and } j = k;$$

$$= 0 \quad \text{if } \left({}^{\alpha}_{\varepsilon}\sigma_j^n\right)_p \text{ or } \left({}^{\alpha}_{\varepsilon}\sigma_j^n\right)_p = 0;$$

otherwise

$$= \frac{E\left\{\left({}^{\alpha}\varepsilon_j^m\right)_p\left({}^{\alpha}\varepsilon_k^n\right)_q\right\}}{\left({}^{\alpha}_{\varepsilon}\sigma_j^m\right)_p \cdot \left({}^{\alpha}_{\varepsilon}\sigma_k^n\right)_p} \quad . \tag{125}$$

Then from Equation (111),

$$ {}^{\alpha}H^{mn}_{jk} = {}^{\alpha}_{\varepsilon}\Sigma^{m}_{j} \cdot {}^{\alpha}_{\varepsilon}R^{mn}_{jk} \cdot {}^{\alpha}_{\varepsilon}\Sigma^{n}_{k} \quad . \tag{126} $$

As before, this representation separates the standard deviations of the components of ${}^{\alpha}\epsilon^{m}_{j}$ and ${}^{\alpha}\epsilon^{n}_{k}$ from correlations among them. If the standard deviations of residual bias do not change with time,

$$ {}^{\alpha}_{\varepsilon}\Sigma^{m}_{j} = {}^{\alpha}_{\varepsilon}\Sigma^{n}_{k} \quad . \tag{127} $$

However, this may not be the case, as when atmospheric refraction corrections are imperfect at low elevation angles. Then ${}^{\alpha}_{\varepsilon}\Sigma^{m}_{j}$ is a function of ${}^{\alpha}q^{m}_{j}$, and Equation (127) is only an approximation.

The form of the correlation matrix ${}^{\alpha}_{\varepsilon}R^{mn}_{jk}$ will depend upon the sensor. Its elements will tend to diminish for large time separations $\left| {}^{\alpha}\hat{t}^{m}_{j} - {}^{\alpha}\hat{t}^{n}_{k} \right|$. For small time separations one may expect that

$$ {}^{\alpha}_{\varepsilon}R^{mn}_{jk} \cong {}^{\alpha}_{\varepsilon}R^{mm}_{jj} \quad . \tag{128} $$

If the components of ${}^{\alpha}\epsilon^{m}_{j}$ are uncorrelated among themselves, then ${}^{\alpha}_{\varepsilon}R^{mn}_{jk}$ will be diagonal and ${}^{\alpha}_{\varepsilon}R^{mm}_{jj}$ will be the identity matrix.

2. Further Structure

The purpose here is to introduce further assumptions about the structure of D that promote ease of inversion. All, some, or none of these further assumptions may be valid for a given problem.

a. Occasionally Decoupled Passes

Suppose for sensor α there exists some time interval ${}^{\alpha}\tau_{zero}$, which is the minimum separation between observations over which residual-bias error correlations vanish:

$$ {}^{\alpha}_{\varepsilon}R^{mn}_{jk} = 0, \; |{}^{\alpha}\hat{t}^{m}_{j} - {}^{\alpha}\hat{t}^{n}_{k}| \geq {}^{\alpha}\tau_{zero} \quad . \tag{129} $$

Then by Equations (114) and (126), a condition on the general observation partition of D is

$$ {}^{\alpha\beta}D^{mn}_{jk} = 0, \; |{}^{\alpha}\hat{t}^{m}_{j} - {}^{\alpha}\hat{t}^{n}_{k}| \geq 0 \quad . \tag{130} $$

Of interest here is the case where the separation between passes by sensor α occasionally exceeds ${}^{\alpha}\tau_{zero}$. (That is, the assumption is not made that every α-pass separation is larger than ${}^{\alpha}\tau_{zero}$.)

To arrive at an appropriate mathematical formulation of such a situation, define an index L = 1,2,..., which counts pass separations for which

$$ {}^{\alpha}\hat{t}^{m+1}_{1} - {}^{\alpha}t^{m} \geq {}^{\alpha}\tau_{zero} \quad . \tag{131} $$

Here ${}^{\alpha}\hat{t}^{m+1}$ is the first observational epoch of pass m+1 for sensor α, and ${}^{\alpha}t^{m}$ is defined to be the last observational epoch of pass m.

Such pass separations decompose the α-pass sequence into a sequence of "multiplets," each comprising one or more passes. Hence one may consider L to be the index of decoupled pass multiplets.

Now L is an index dependent upon α and m:

$$L = L(\alpha, m) \quad . \tag{132a}$$

One may express the functional dependence upon m recursively. Let

$$L(\alpha, 1) \equiv 1 \quad . \tag{132b}$$

For arbitrary m, if Equation (131) holds

$$L(\alpha, m+1) = L(\alpha, m) + 1 \quad ; \tag{133a}$$

otherwise

$$L(\alpha, m+1) = L(\alpha, m) \quad . \tag{133b}$$

Now let

$$^{\alpha L}H_{jk}^{mn} \equiv E\left\{ {}^{\alpha}\epsilon_j^m \; {}^{\alpha}\epsilon_k^{n\dagger} \right\} \quad , \tag{134a}$$

under the constraint that passes m and n are in the same multiplet:

$$L(\alpha, m) = L(\alpha, n) \quad . \tag{134b}$$

The assumption of occasionally decoupled passes is just

$$^{\alpha}H_{jk}^{mn} = \delta_{L(\alpha, m), L(\alpha, n)} \; {}^{\alpha L}H_{jk}^{mn} \quad . \tag{135}$$

[see Equations (126) and (129)].

Utilizing the redundancy

$$\delta_{mn} = \delta_{L(\alpha, m), L(\alpha, n)} \; \delta_{mn} \quad , \tag{136}$$

one may now write Equation (114), subsuming Equation (130), as

$$ {}^{\alpha\beta}D^{mn}_{jk} = \delta_{\alpha\beta}\, \delta_{L(\alpha,\, m),\, L(\alpha,\, n)} \left({}^{\alpha}G^{m}_{j}\, \delta_{mn}\, \delta_{jk} + {}^{\alpha L}H^{mn}_{jk} \right) \quad . \tag{137} $$

Thus D is block diagonal. If the blocks of D are denoted ${}^{\alpha L}D$, then the problem of inverting D reduces to that of inverting the set of smaller matrices ${}^{\alpha L}D$.

b. Strongly Coupled Passes

Suppose for each sensor α there exists some maximum time interval ${}^{\alpha}\tau_{\text{one}}$, for which residual-bias error standard deviations and correlations are invariant—i.e., Equations (127) and (128) hold when

$$ \left| {}^{\alpha}\hat{t}^{m}_{j} - {}^{\alpha}\hat{t}^{n}_{k} \right| \leq {}^{\alpha}\tau_{\text{one}} \quad . \tag{138} $$

In particular, suppose ${}^{\alpha}\tau_{\text{one}}$ is greater than the duration of any pass multiplet as defined in the previous subsection. Then Equation (138) holds whenever

$$ L(\alpha,\, m) = L(\alpha,\, n) \tag{139} $$

and over each multiplet one has an invariant matrix

$$ {}^{\alpha L}H \equiv {}^{\alpha L}H^{mn}_{jk} \quad . \tag{140} $$

One may refer to this as the assumption of *strongly coupled passes*.

A necessary, but not necessarily sufficient, condition for the existence of multiplets that are each strongly coupled internally, yet are decoupled from one multiplet to another, is that

$$ {}^{\alpha}\tau_{\text{one}} \ll {}^{\alpha}\tau_{\text{zero}} \quad . \tag{141} $$

For example, if ${}^{\alpha}\tau_{zero}$ happens to be one satellite period, then each multiplet can comprise but one pass. For a low-altitude satellite, if ${}^{\alpha}\tau_{zero} \sim 12$ hours, then each multiplet might consist of a pair of passes on successive revolutions. For some problems, Equation (141) cannot be satisfied for multiplets.

When, however, Equation (140) is valid, substitution into Equation (137) yields

$$ {}^{\alpha\beta}D^{mn}_{jk} = \delta_{\alpha\beta}\ \delta_{L(\alpha,\, m),\, L(\alpha,\, n)}\left({}^{\alpha}G^{m}_{j}\ \delta_{mn}\ \delta_{jk} + {}^{\alpha}L_{H}\right) \quad . \tag{142} $$

When D has this structure, the general observational partition of its inverse turns out to be available analytically:

$$ {}^{\alpha\beta}\left(D^{-1}\right)^{mn}_{jk} = \delta_{\alpha\beta}\ \delta_{L(\alpha,\, m),\, L(\alpha,\, n)} $$

$$ \cdot\left[\left({}^{\alpha}G^{m}_{j}\right)^{-1} \delta_{mn}\ \delta_{jk} - \left({}^{\alpha}G^{m}_{j}\right)^{-1}\ {}^{\alpha}L_{U}\ {}^{\alpha}L_{H}\left({}^{\alpha}G^{n}_{k}\right)^{-1}\right] \quad , \tag{143} $$

where

$$ {}^{\alpha}L(\alpha,\, m)_{U} \equiv \left[1 + {}^{\alpha}L_{H} \sum_{\ell,\, i} \delta_{L(\alpha,\, m)\, L(\alpha,\, \ell)}\left({}^{\alpha}G^{\ell}_{i}\right)^{-1}\right]^{-1} \quad . \tag{144} $$

These expressions for ${}^{\alpha\beta}\left(D^{-1}\right)^{mn}_{jk}$ constitute a generalization of the results of Appendix C. That appendix derives the inverse of a matrix having its general partition of the same form as

$$ {}^{\alpha}G^{m}_{j}\ \delta_{mn}\ \delta_{jk} + {}^{\alpha}L_{H} $$

[see Equation (142)], but with the restriction that ${}^{\alpha}G_j^m$ and ${}^{\alpha}L_H$ must be diagonal. That restriction allows those two matrices to commute.

However, the result of Equations (143) and (144) embodies no such restriction, as one may show by proving that those Equations yield the proper result for an inverse:

$$\sum_{\gamma \ell i} {}^{\alpha\gamma}D_{ji}^{m\ell} \cdot {}^{\alpha\beta}\left(D^{-1}\right)_{ik}^{\ell n} = \delta_{\alpha\beta}\,\delta_{mn}\,\delta_{jk} \quad . \tag{145}$$

The proof proceeds by direct substitution of Equations (136), (142), and (143) into the lefthand side here:

$$\begin{aligned}
&\sum_{\gamma \ell i} \delta_{\alpha\gamma}\, {}^{\alpha}G_j^m\, \delta_{m\ell}\, \delta_{ji} \cdot \delta_{\gamma\beta} \left({}^{\gamma}G_i^{\ell}\right)^{-1} \delta_{\ell n}\, \delta_{ik} \\
&- \sum_{\gamma \ell i} \delta_{\alpha\gamma}\, {}^{\alpha}G_j^m\, \delta_{m\ell}\, \delta_{ji} \cdot \delta_{\gamma\beta}\, \delta_{L(\gamma,\ell),\,L(\beta,n)} \left({}^{\gamma}G_i^{\ell}\right)^{-1} {}^{\alpha}L_U\, {}^{\alpha}L_H \left({}^{\beta}G_k^n\right)^{-1} \\
&+ \sum_{\gamma \ell i} \delta_{\alpha\gamma}\, \delta_{L(\alpha,m),\,L(\gamma,\ell)}\, {}^{\alpha}L_H \cdot \delta_{\gamma\beta} \left({}^{\gamma}G_k^{\ell}\right)^{-1} \delta_{\ell n}\, \delta_{ik} \\
&- \sum_{\gamma \ell i} \delta_{\alpha\gamma}\, \delta_{L(\alpha,m),\,L(\gamma,\ell)}\, {}^{\alpha}L_H \cdot \delta_{\gamma\beta}\, \delta_{L(\gamma,\ell),\,L(\beta,n)} \left({}^{\gamma}G_i^{\ell}\right)^{-1} {}^{\alpha}L_U\, {}^{\alpha}L_H \left({}^{\beta}G_k^n\right)^{-1} \\
&= \delta_{\alpha\beta}\, \delta_{mn}\, \delta_{jk} \\
&- \delta_{\alpha\beta}\, \delta_{L(\alpha,m),\,L(\beta,n)} \cdot {}^{\alpha}L_U \cdot {}^{\alpha}L_H \cdot \left({}^{\beta}G_k^n\right)^{-1} \\
&+ \delta_{\alpha\beta}\, \delta_{L(\alpha,m),\,L(\alpha,n)} \cdot {}^{\alpha}L_H \cdot \left({}^{\alpha}G_k^n\right)^{-1} \\
&- \delta_{\alpha\beta} \left(\sum_{\ell i} \delta_{L(\alpha,m),\,L(\alpha,\ell)} \cdot {}^{\alpha}L_H \cdot \delta_{L(\alpha,\ell),\,L(\alpha,n)} \cdot \left({}^{\alpha}G_i^{\ell}\right)^{-1} \cdot {}^{\alpha}L_U \cdot {}^{\alpha}L_H \right) \cdot \\
&\qquad \cdot \left({}^{\beta}G_k^n\right)^{-1} \quad .
\end{aligned} \tag{146}$$

The last term here is a sum over just the multiplet L, and has no nonvanishing terms over that multiplet. Since ${}^{\alpha}L_H$ and ${}^{\alpha}L_U$ are constants over the multiplet, one can factor them from under the summation. Moreover, one can see (via a "truth table" analysis) that

$$\delta_{L(\alpha,m),L(\alpha,\ell)}\,\delta_{L(\alpha,\ell),L(\alpha,n)} = \delta_{L(\alpha,m)\,L(\alpha,n)}\,\delta_{L(\alpha,m)\,L(\alpha,\ell)} \cdot \quad (147)$$

Hence one may continue to develop Equation (146) as

$$\begin{aligned}
&= \delta_{\alpha\beta}\,\delta_{mn}\,\delta_{jk} \\
&\quad + \delta_{\alpha\beta}\,\delta_{L(\alpha,m)\,L(\alpha,n)}\Big[-\,{}^{\alpha}L_U\,{}^{\alpha}L_H + {}^{\alpha}L_H + \\
&\qquad -\,{}^{\alpha}L_H\Big(\sum_{\ell i}\delta_{L(\alpha,m),L(\alpha,\ell)}\big({}^{\alpha}G_i^{\ell}\big)^{-1}\Big)\cdot{}^{\alpha}L_U\cdot{}^{\alpha}L_H\Big]\big({}^{\alpha}G_k^{n}\big)^{-1} \\
&= \delta_{\alpha\beta}\,\delta_{mn}\,\delta_{jk} \\
&\quad + \delta_{\alpha\beta}\,\delta_{L(\alpha,m)\,L(\alpha,n)}\Big[{}^{\alpha}L_H + \\
&\qquad -\Big(1 + {}^{\alpha}L_H\cdot\sum_{\ell i}\delta_{L(\alpha,m)\,L(\alpha,\ell)}\big({}^{\alpha}G_i^{\ell}\big)^{-1}\Big)\cdot{}^{\alpha}L_U\cdot{}^{\alpha}L_H\Big]\big({}^{\alpha}G_k^{n}\big)^{-1} \qquad (148) \\
&= \delta_{\alpha\beta}\,\delta_{mn}\,\delta_{jk} \quad ,
\end{aligned}$$

as required. The last step above resulted from substitution of Equation (144) for ${}^{\alpha}L_U$.

With regard to Equation (143), note that D^{-1} reduces to a form block diagonal in $\left({}^{\alpha}G_j\right)^{-1}$ if all the ${}^{\alpha}L_H$ happen to vanish. Thus, one may regard D^{-1} as the difference between a "noise" covariance matrix and a "residual bias" correction matrix, respectively the first and second terms of Equation (143).

D. Some Expansions of the General Solution

The calculational efficiency of state-vector estimation depends upon the degree of difficulty of extracting C^{-1} in the estimator W_i [see Equation (70)]. In "least squares" estimation, C is taken to be diagonal—or at least block diagonal, where each block dimension is of the order of the observation-vector dimension. Such a C may be very different from D, if the latter contains extensive correlation terms among residual-bias errors. Then resulting ephemeris errors will be worse than the optimum (minimum-variance) errors when $C = D$.

Of course, one may choose C to be a better approximation to a highly correlated D, but generally at significant cost in calculational practicality.

The purpose here is to find explicit, detailed representations for S_i, the covariance matrix of state-vector errors, for various combinations of some possible structures for C and D. Each S_i representation will exhibit a characteristic level of approximation to the minimum-variance form (when $C = D$) and will also afford a characteristic level of calculational efficiency not only for W_i, but also for S_i.

Structures of D considered here derive from the previous section. Structures for C considered here also derive from the previous section, but will occur in a distinctive notation in order to avoid confusion with D.

In the interest of notational simplicity, the ensuing expansions of Equation (100) will omit the subscript i. The quantities S, W, and T are still to be understood as corresponding to the estimation epoch $\hat{t}_i$. The indexing convention of the expansions will be the same as in Section III.C, i.e., it will correspond to the observational epochs ${}^{\alpha}\hat{t}_j^m$.

Results will occur as formulas for X^{-1} and Z, where

$$X \equiv \left(T^{\dagger} C^{-1} T\right)^{-1} \tag{149}$$

and

$$Z \equiv \left(C^{-1} T\right)^{\dagger} D\left(C^{-1} T\right) \quad . \tag{150}$$

Then, taking into account the symmetry of C,

$$S = X Z X \quad . \tag{151}$$

Also,

$$W = X\left(C^{-1} T\right) \quad . \tag{152}$$

Note that if C = D, then S = X. Thus X is the covariance matrix of state-vector errors for the case of optimal, minimum-variance estimation. Accordingly, ZX (or XZ) is the matrix factor on X by which S falls short of optimality.

Numeric inversion of X^{-1} to yield X is not ordinarily a significant calculational problem, since X is only a 6 x 6 matrix and need be inverted only occasionally [see discussion of Section II.E and Equation (102)]. Hence, one may regard the calculational difficulty of finding W and S as effectively as that of finding X^{-1} and Z—the latter of course containing $C^{-1}T$ [see Equations (150) and (152)].

Table 5 gives seven combinations of structure assumptions for C and D, along with appropriate analytic expressions for C^{-1}. Table 6 gives corresponding formulas for X^{-1} and Z.

The notes in Table 5 pertain, first, to calculational ease of finding X^{-1} and Z, given a set of measurement data; and second, to ease of recalculating X^{-1} and Z as new measurement data become available. Additional comments upon each case will conclude this section.

Case 1 imposes no restrictions upon C or D, except that all error correlations vanish from one sensor to another. These matrices are then block diagonal, respectively, in partitions ${}^{\alpha}C$ and ${}^{\alpha}D$. The formulas for X^{-1} and Z are algebraically simple but calculationally complex, requiring numeric inversion of the large partitions ${}^{\alpha}C$.

Case 2 introduces the structure of Equation (114) for D, and a similar structure for C in terms of ${}^{\alpha}E_j^m$ (analogous to ${}^{\alpha}G_j^m$) and ${}^{\alpha}F_{jk}^{mn}$ (analogous to ${}^{\alpha}H_{jk}^{mn}$). These structures improve the physical interpretability of C and D, but are generally still not calculationally practical in that inversion of each of the large partitions ${}^{\alpha}C$ is still necessary.

Case 3 reduces C to a form block diagonal in ${}^{\alpha}E_j^m$, resulting in a least-squares form of W. However, careful inspection of the expression for Z in Case 3 will reveal that the generality of the D-structure places significant demands upon computer memory in recalculating Z as new data become available.

TABLE 5
SOME STRUCTURE COMBINATIONS FOR C AND D

Case	D: Item	D: Partition	C: Item	C: Partition	C: Item Inverse Partition	KEY: o Error correlation assumptions; o (1) Matrix to be inverted; (2) Matrix dimension; (3) Number of inversions required; o Comments
1	D	${}^{\alpha}D\delta_{\alpha\beta}$	C	${}^{\alpha}C\delta_{\alpha\beta}$	${}^{\alpha}C^{-1}\delta_{\alpha\beta}$	o No sensor-to-sensor correlations o (1) ${}^{\alpha}C$; (2) Total α-observations, times no. of measured components per obsn.; (3) Number of sensors o A minimum-variance approach, very complex both calculationally and in specification of C and D.
2	${}^{\alpha}D$	${}^{\alpha}G_j^m\delta_{mn}\delta_{jk} + {}^{\alpha}H_{jk}^{mn}$	${}^{\alpha}C$	${}^{\alpha}E_j^m\delta_{mn}\delta_{jk} + {}^{\alpha}F_{jk}^{mn}$	$\left({}^{\alpha}C^{-1}\right)_{jk}^{mn}$	o Same as Case 1, but with separated types of errors (see IIIC-1) o Same as Case 1 o Same as Case 1, but improved ease of interpretation
3	${}^{\alpha}D$	(Same as Type 3)	${}^{\alpha}C$	${}^{\alpha}E_j^m\delta_{mn}\delta_{jk}$	$\left({}^{\alpha}E_j^m\right)^{-1}\delta_{mn}\delta_{jk}$	o Same as Case 1 for D, but zero residual biases for C o (1) ${}^{\alpha}E_j^m$; (2) No. of measured components per observation; (3) Number of observations by *all sensors* o At least-squares approach, calculationally feasible but with some *drawbacks because* of the generality of D
4	${}^{\alpha}D$	${}^{\alpha L}D\,\delta_{L(\alpha,m)\,L(\alpha,n)}$	${}^{\alpha}C$	${}^{\alpha L}C\,\delta_{L(\alpha,m)\,L(\alpha,n)}$	${}^{\alpha L}C^{-1}\,\delta_{L(\alpha,m)\,L(\alpha,n)}$	o Same as Case 1, but also no inter-multiplet correlations (see IIIC-2a) o (1) ${}^{\alpha L}C$; (2) Total number of observations in multiplet times no. of measured components per obsn. o A minimum-variance approach, calculationally better than Case 1 but still very complex
5	${}^{\alpha L}D$	${}^{\alpha L}G_j^m\delta_{mn}\delta_{jk} + {}^{\alpha L}H_{jk}^{mn}$	${}^{\alpha L}C$	${}^{\alpha L}E_j^m\delta_{mn}\delta_{jk} + {}^{\alpha L}F_{jk}^{mn}$	$\left({}^{\alpha L}C^{-1}\right)_{jk}^{mn}$	o Same as Case 4, but with separated types of errors o Same as Case 4 o Same as Case 4, but with improved ease of interpretation.
6	${}^{\alpha L}D$	(Same as Case 5)	${}^{\alpha L}C$	${}^{\alpha L}E_j^m\delta_{mn}\delta_{jk} + {}^{\alpha L}F$	$\left({}^{\alpha L}E_j^m\right)^{-1}\delta_{mn}\delta_{jk} + -\left({}^{\alpha L}E_j^m\right)^{-1}{}^{\alpha L}g\,{}^{\alpha L}F\left({}^{\alpha L}E_k^n\right)^{-1}$, ${}^{\alpha L}g = \left[1 + {}^{\alpha L}F\sum_{m,j}\left({}^{\alpha L}E_j^m\right)^{-1}\right]^{-1}$	o For D, same as Case 5; but for C, constant residual biases within a multiplet o (1) ${}^{\alpha L}E_j^m$; (2) Number of measured components per obsn.; (3) Number of obsns., all sensors o A minimum-variance approach, *calculationally feasible*
7	${}^{\alpha L}D$	${}^{\alpha L}G_j^m\delta_{mn}\delta_{jk} + {}^{\alpha L}H$	${}^{\alpha L}C$	(Same as Case 6)	(Same as Case 6)	o For both C and D, constant residual biases within a multiplet o Same as Case 5 o Same as Case 5, *but easy* to update as new data become available

TABLE 6
STATE-VECTOR ERROR MATRICES
FOR STRUCTURE OF TABLE 5

Case	S = XZX	
	X^{-1}	Z
1	$\sum_\alpha \sum_{m,j}\sum_{n,k} \left({}^{\alpha}T_j^{m\dagger} ({}^{\alpha}C^{-1})_{jk}^{mn}\, {}^{\alpha}T_k^n\right)$	$\sum_\alpha\sum_{m,j}\sum_{n,k}\left[\left(\sum_{p,i}({}^\alpha C^{-1})_{ji}^{mp}\,{}^\alpha T_i^p\right)^\dagger {}^\alpha D_{jk}^{mn}\left(\sum_{q,\ell}({}^\alpha C^{-1})_{k\ell}^{nq}\,{}^\alpha T_\ell^q\right)\right]$
2	(Same as Case 1)	$\sum_\alpha\sum_{m,j}\left[\left(\sum_{p,i}({}^\alpha C^{-1})_{ji}^{mp}\,{}^\alpha T_i^p\right)^\dagger {}^\alpha G_j^m \left(\sum_{q,\ell}({}^\alpha C^{-1})_{j\ell}^{mq}\,{}^\alpha T_\ell^q\right)\right] +$ $+ \sum_\alpha\sum_{m,j}\sum_{n,k}\left[\left(\sum_{p,i}({}^\alpha C^{-1})_{ji}^{mp}\,{}^\alpha T_i^p\right)^\dagger {}^\alpha H_{jk}^{mn}\left(\sum_{q,\ell}({}^\alpha C^{-1})_{k\ell}^{nq}\,{}^\alpha T_\ell^q\right)\right]$
3	$\sum_\alpha\sum_{m,j}\left({}^\alpha T_j^{m\dagger}({}^\alpha E_j^m)^{-1}\,{}^\alpha T_j^m\right)$	$\sum_\alpha\sum_{m,j}\left[\left(({}^\alpha E_j^m)^{-1}\,{}^\alpha T_j^m\right)^\dagger {}^\alpha G_j^m\left(({}^\alpha E_j^m)^{-1}\,{}^\alpha T_j^m\right)\right] +$ $+ \sum_\alpha\sum_{m,j}\sum_{n,k}\left[\left(({}^\alpha E_j^m)\,{}^\alpha T_j^m\right)^\dagger {}^\alpha H_{jk}^{mn}\left(({}^\alpha E_k^n)^{-1}\,{}^\alpha T_k^n\right)\right]$
4	$\sum_\alpha\sum_L\sum_{m,j}^{(L)}\sum_{n,k}^{(L)}\left({}^{\alpha L}T_j^{m\dagger}\,({}^{\alpha L}C^{-1})_{jk}^{mn}\,{}^{\alpha L}T_k^n\right)$	$\sum_\alpha\sum_L\sum_{m,j}^{(L)}\sum_{n,k}^{(L)}\left[\left(\sum_{p,i}^{(L)}({}^{\alpha L}C^{-1})_{ji}^{mp}\,{}^{\alpha L}T_i^p\right)^\dagger {}^{\alpha L}D_{jk}^{mn}\left(\sum_{q,\ell}^{(L)}({}^{\alpha L}C^{-1})_{k\ell}^{nq}\,{}^{\alpha L}T_\ell^q\right)\right]$
5	(Same as Case 4)	$\sum_\alpha\sum_L\sum_{m,j}^{(L)}\left[\left(\sum_{p,i}^{(L)}({}^{\alpha L}C^{-1})_{ji}^{mp}\,{}^{\alpha L}T_i^p\right)^\dagger {}^{\alpha L}G_j^m\left(\sum_{q,\ell}^{(L)}({}^{\alpha L}C^{-1})_{j\ell}^{mq}\,{}^{\alpha L}T_\ell^q\right)\right] +$ $+ \sum_\alpha\sum_L\sum_{m,j}^{(L)}\sum_{n,k}^{(L)}\left[\left(\sum_{p,i}^{(L)}({}^{\alpha L}C^{-1})_{ji}^{mp}\,{}^{\alpha L}T_i^p\right)^\dagger {}^{\alpha L}H_{jk}^{mn}\left(\sum_{q,\ell}^{(L)}({}^{\alpha L}C^{-1})_{k\ell}^{nq}\,{}^{\alpha L}T_\ell^q\right)\right]$
6	$\sum_\alpha\sum_L\left[\sum_{m,j}^{(L)}\left({}^{\alpha L}T_j^{m\dagger}({}^{\alpha L}E_j^m)^{-1}\,{}^{\alpha L}T_j^m\right) - {}^{\alpha L}V^\dagger\,{}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right],$ ${}^{\alpha L}V \equiv \sum_{m,j}^{(L)}\left(({}^{\alpha L}E_j^m)^{-1}\,{}^{\alpha L}T_j^m\right)$	$\sum_\alpha\sum_L\sum_{m,j}^{(L)}\left[\left({}^{\alpha L}T_j^m - {}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)^\dagger({}^{\alpha L}E_j^m)^{-1}\,{}^{\alpha L}G_j^m({}^{\alpha L}E_j^m)^{-1}\left({}^{\alpha L}T_j^m - {}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)\right] +$ $+ \sum_\alpha\sum_L\sum_{m,j}^{(L)}\sum_{n,k}^{(L)}\left[\left({}^{\alpha L}T_j^m - {}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)^\dagger\left({}^{\alpha L}E_j^m\right)^{-1}{}^{\alpha L}H_{jk}^{mn}\left({}^{\alpha L}E_k^n\right)^{-1}\left({}^{\alpha L}T_k^n - {}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)\right]$
7	(Same as Case 6)	$\sum_\alpha\sum_L\sum_{m,j}^{(L)}\left[\left({}^{\alpha L}T_j^m - {}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)^\dagger({}^{\alpha L}E_j^m)^{-1}\,{}^{\alpha L}G_j^m\,({}^{\alpha L}E_j^m)^{-1}\left({}^{\alpha L}T_j^m - {}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)\right] +$ $+ \sum_\alpha\sum_L\left[\left({}^{\alpha L}V - \sum_{m,j}^{(L)}({}^{\alpha L}E_j^m)^{-1}\,{}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)^\dagger {}^{\alpha L}K\left({}^{\alpha L}V - \sum_{m,j}^{(L)}({}^{\alpha L}E_j^m)^{-1}\,{}^{\alpha L}U\,{}^{\alpha L}F\,{}^{\alpha L}V\right)\right]$

Case 4 begins a different type of structuring, relative to Case 1 as a baseline. A pass-multiplet structure for C and D substitutes the problem of inverting the partitions ${}^{\alpha L}C$ rather than the ${}^{\alpha}C$. The ${}^{\alpha L}C$ tend, however, to still be of large dimension.

Case 5 is the pass-multiplet analog of Case 2 and is calculationally not greatly advantageous to Case 2.

Case 6 introduces a structure for C analogous to Equation (142) for D. The inverse of this structure exists in simple analytic form and makes calculation of C^{-1} only slightly more involved than for the least-squares Case 3. Thus, the inversion Equations (143) and (144) effectively afford a limited, calculationally efficient generalization of least squares estimation.

With Case 6 as with Case 3, however, the general form of D does impose some calculational penalties in regard to Z as new data become available.

In Case 7, both C and D have the structure of Equation (142). Calculation of Z is thereby considerably simpler. (Further specializations of Case 7 will appear in the next section.)

E. <u>The Computer Model SEEM</u>

The purpose of this section is to describe the analytical basis of the computer model SEEM (see Chapter I). A further purpose is to provide some numerical outputs of SEEM as examples of calculational results obtainable using the analytical formalism of this report.

1. Analytical Basis of SEEM

The approach here is the further specialization of Case 7 of the previous section.

For D, assume that each pass multiplet is a single pass. Further, assume that ${}^{\alpha}L_H$ is the same for every multiplet. Then one may write

$$ {}^{\alpha\beta}D^{mn}_{jk} = \delta_{\alpha\beta}\,\delta_{mn}\left(\delta_{jk}\;{}^{\alpha}G^{m}_{j} + {}^{\alpha}H\right) \quad . \tag{153} $$

Make further assumptions about the internal structures of ${}^{\alpha}G^{m}_{j}$ and ${}^{\alpha}H$. Assume that within any given observation the noise errors are uncorrelated with each other and the residual bias errors are uncorrelated with each other. Then both ${}^{\alpha}_{\nu}R^{m}_{j}$ [see Equation (121b)] and ${}^{\alpha}_{\varepsilon}R^{mn}_{jk}$ [see Equation (125)] are identity matrices.

Then from Equations (119) and (122), the general element of ${}^{\alpha}G^{m}_{j}$ is

$$ \left({}^{\alpha}G^{m}_{j}\right)_{pq} = \delta_{pq}\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)^{2}_{p} \quad . \tag{154} $$

From Equations (123) and (126), the general element of ${}^{\alpha}H$ is

$$ \left({}^{\alpha}H\right)_{pq} = \delta_{pq}\left({}^{\alpha}_{\varepsilon}\sigma\right)^{2}_{p} \quad . \tag{155} $$

SEEM at present provides for just one sensor type, radars operating in altazimuth coordinates (see Section II.A).* The noise-variance components of Equation (154) become

$$\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)^2_1 = \left({}^{\alpha}_{\nu}\sigma_{\chi}\right)^2 \cdot (1/\cos h)^2 \quad ; \tag{156}$$

$$\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)^2_2 = \left({}^{\alpha}_{\nu}\sigma_{h}\right)^2 \quad ; \tag{157}$$

$$\left({}^{\alpha}_{\nu}\sigma^{m}_{j}\right)^2_3 = \left({}^{\alpha}_{\nu}\sigma_{\rho}\right)^2 \quad . \tag{158}$$

Here χ, h, and ρ respectively denote azimuth, elevation, and range, and the symbol σ on the righthand side represents a constant for a given radar. The factor $(1/\cos h)^2$ in Equation (156) accounts for the loss in azimuthal accuracy at high elevation angles.

SEEM assumes the following for the residual-bias variances of Equation (155):

$$\left({}^{\alpha}_{\varepsilon}\sigma\right)^2_1 = \left({}^{\alpha}_{\varepsilon}\sigma_{\chi}\right)^2 \quad ; \tag{159}$$

$$\left({}^{\alpha}_{\varepsilon}\sigma\right)^2_2 = \left({}^{\alpha}_{\varepsilon}\sigma_{h}\right)^2 \quad ; \tag{160}$$

$$\left({}^{\alpha}_{\varepsilon}\sigma\right)^2_3 = \left({}^{\alpha}_{\varepsilon}\sigma_{\rho}\right)^2 \quad . \tag{161}$$

---

* One may also, by an input strategem, make SEEM accommodate altazimuth-coordinate telescopes. The strategem is to assign a very large value to ${}^{\alpha}_{\nu}\sigma_{\rho}^2$ of Equation (158), thereby assigning range measurements negligible statistical weight.

Again the σ symbols on the righthand side are constants for a given radar.

As regards C, SEEM provides two options. The first, a "least squares" option, is

$$
{}^{\alpha\beta}\left(C_{LS}\right)^{mn}_{jk} = \delta_{\alpha\beta}\,\delta_{mn}\,\delta_{jk}\,{}^{\alpha}G^{m}_{j} \quad . \tag{162}
$$

The second, a "minimum variance" option, is

$$
{}^{\alpha\beta}\left(C_{MV}\right)^{mn}_{jk} = {}^{\alpha\beta}D^{mn}_{jk} \quad . \tag{163}
$$

Appropriate manipulation of Case 7 results of the preceding section then yields for the least squares option

$$
S_{LS} = X_{LS} + X_{LS}\left[\sum_{\alpha}\sum_{m}\left({}^{\alpha}V^{m\dagger}\ {}^{\alpha}H\ {}^{\alpha}V^{m}\right)\right]X_{LS} \quad , \tag{164}
$$

where

$$
X^{-1}_{LS} = \sum_{\alpha}\ \sum_{m}\left(\sum_{j}\ {}^{\alpha}T^{m\dagger}_{j}\left({}^{\alpha}G^{m}_{j}\right)^{-1}\ {}^{\alpha}T^{m}_{j}\right) \tag{165}
$$

and

$$
{}^{\alpha}V^{m} \equiv \sum_{j}\left({}^{\alpha}G^{m}_{j}\right)^{-1}\ {}^{\alpha}T^{m}_{j} \quad . \tag{166}
$$

For the minimum-variance option,

$$
S_{MV} = X_{MV} \quad , \tag{167}
$$

where

$$X_{MV}^{-1} = \sum_{\alpha} \sum_{m} \left(\sum_{j} {}^{\alpha}T_j^{m\dagger} \left({}^{\alpha}G_j^m\right)^{-1} {}^{\alpha}T_j^m \right) + \sum_{\alpha} \sum_{m} \left({}^{\alpha}V^{m\dagger} \, {}^{\alpha}U^m \, {}^{\alpha}H \, {}^{\alpha}V^m \right) \tag{168}$$

and

$${}^{\alpha}U^m \equiv \left[1 + {}^{\alpha}H \sum_{j} \left({}^{\alpha}G_j^m\right)^{-1} \right]^{-1} \quad . \tag{169}$$

Evaluation of the ${}^{\alpha}T_j^m$ in these expressions is via Equations (98), since T_j there was redesignated at ${}^{\alpha}T_j^m$ in the notational change of Subsection III.C.1. On the righthand side of Equation (98a), the M_j is the "radar" entry of Table 4. The Q_j^{-1} are those of Table 1, as specialized in Table 2. Appendix B gives expressions for Φ_{ji} and Ψ_{ji} of Equation (98b).

Thus, SEEM evaluates S_{LS} or S_{MV} as S_i at an appropriate time $\hat{t}_i$. Equation (102) then yields S_k at requisite prediction times $\hat{t}_k$, $k = 1, 2, \ldots n$. For each $\hat{t}_k$, SEEM extracts ${}^{r}S_k$ from S_i [see defining Equation (91)] and then obtains $\left[{}^{r}S_k\right]_{UVW}$ via Equation (103), evaluating the rotation matrix L as developed in Section II.A.

The diagonal elements of $\left[{}^{r}S_k\right]_{UVW}$ are the variances σ_{kU}^2, σ_{kV}^2, σ_{kW}^2, respectively the radial, along-track, and cross-track component variances of the ephemeris vector at $\hat{t}_k$. The resultant-vector variance is

$$\sigma_k^2 = \sigma_{Uk}^2 + \sigma_{Vk}^2 + \sigma_{Wk}^2 \quad . \tag{170}$$

SEEM outputs $3\sigma_k$, $3\sigma_{kU}$, $3\sigma_{kV}$, and $3\sigma_{kW}$ for the prediction times $\hat{t}_k$, $k = 1, 2, \ldots, n$.

As it is now programmed, SEEM does not obtain the principal axes and orientation of the ephemeris error ellipsoid by diagonalizing $\left[{}^rS_k\right]_{UVW}$, etc. However, for near-circular orbits this ellipsoid tends to be oriented along the UVW coordinates, primarily because of the effect of period uncertainty, which makes the along-track error ordinarily large compared to radial and cross-track errors.

Thus, for near-circular orbits one may regard $3\sigma_{kU}$, $3\sigma_{kV}$, and $3\sigma_{kW}$ as approximately the principal dimensions of the 60.8-percent confidence error ellipsoid (see Table A-1). The resultant $3\sigma_k$ is the exact RSS dimension of the ellipsoid, since the trace of $\left[{}^rS_k\right]_{UVW}$ is invariant under the coordinate-frame rotation of diagonalization.

2. Representative Ephemeris Error Results of SEEM

This subsection gives representative graphical outputs of SEEM. All outputs given here correspond to a satellite in circular orbit at an altitude of 400 km. The epoch $\hat{t} = 0$ is that of "launch," when the satellite initially appears in orbit.

Sensors are altazimuth radars, with hemispheric coverage down to a minimum elevation angle of 7°. Maximum range is set, not by radar capability, but rather by horizon line-of-sight cutoff at minimum elevation. Sensors provide measurements at 6-second intervals while the satellite is within coverage. For "nominal" measurements, "noise" and "residual bias" standard deviations are those of Table 7.

SEEM output graphics of Figures 3 through 6 correspond to the specific input conditions of Table 8. Figure 3 represents a pass through radar coverage about 15 minutes after "launch." Error plots commence a few minutes after the satellite exits coverage, simulating a data-processing delay in availability of estimated ephemerides.*

Note that for this case, minimum-variance estimation yields little accuracy improvement over least squares estimation.

The error curves of Figure 3 exhibit several typical features that deserve comment. First, the cumulative growth of along-track error is what one would expect from error in estimating the satellite period. By contrast, the radial and cross-track errors repeat with each satellite revolution. Thus, along-track error soon dominates the resultant error.

Second, all error components contain the satellite period (100 minutes) as a fundamental harmonic, with minima at or near one-period intervals from the radar pass. This behavior is not surprising, since the radar pass corresponds to a point in inertial space where position is actually measured, i.e., where both estimated and true orbits are closest together.

---

* In Figure 3, the width of the "radar coverage rectangle" denotes time in coverage, but the height of the rectangle has no interpretational significance.

TABLE 7
"NOMINAL" STANDARD DEVIATIONS OF MEASUREMENT

Coordinate	Noise	Residual Bias
Azimuth	${}^{\alpha}_{\nu}\sigma_{\chi} = 0.05°$	${}^{\alpha}_{\epsilon}\sigma_{\chi} = 0.05°$
Elevation	${}^{\alpha}_{\nu}\sigma_{h} = 0.05°$	${}^{\alpha}_{\epsilon}\sigma_{\chi} = 0.05°$
Range	${}^{\alpha}_{\nu}\sigma_{p} = 50$ m.	${}^{\chi}_{\epsilon}\sigma_{p} = 50$ m.

TABLE 8
INPUTS FOR SEEM EXAMPLES

Figure Number	Std. Deviations of Measurement		Pass Spacing	Estimation Method
	Noise	Residual Bias		
3 (Solid Curve)	Nominal	Nominal	—	Least Squares
3 (Broken Curve)	Nominal	Nominal	—	Minimum Variance
4	Nominal	Zero	—	LS and MV (Same Curve)
5	Nominal*	Nominal*	1/4 Revolution	Least Squares
6	Nominal*	Nominal*	1/2 Revolution	Least Squares

* Applies to both radars

FIGURE 3
EPHERMIS ERRORS:
SINGLE-PASS, "NOMINAL" MEASUREMENT ERRORS
(Facsimile of Computer Output)

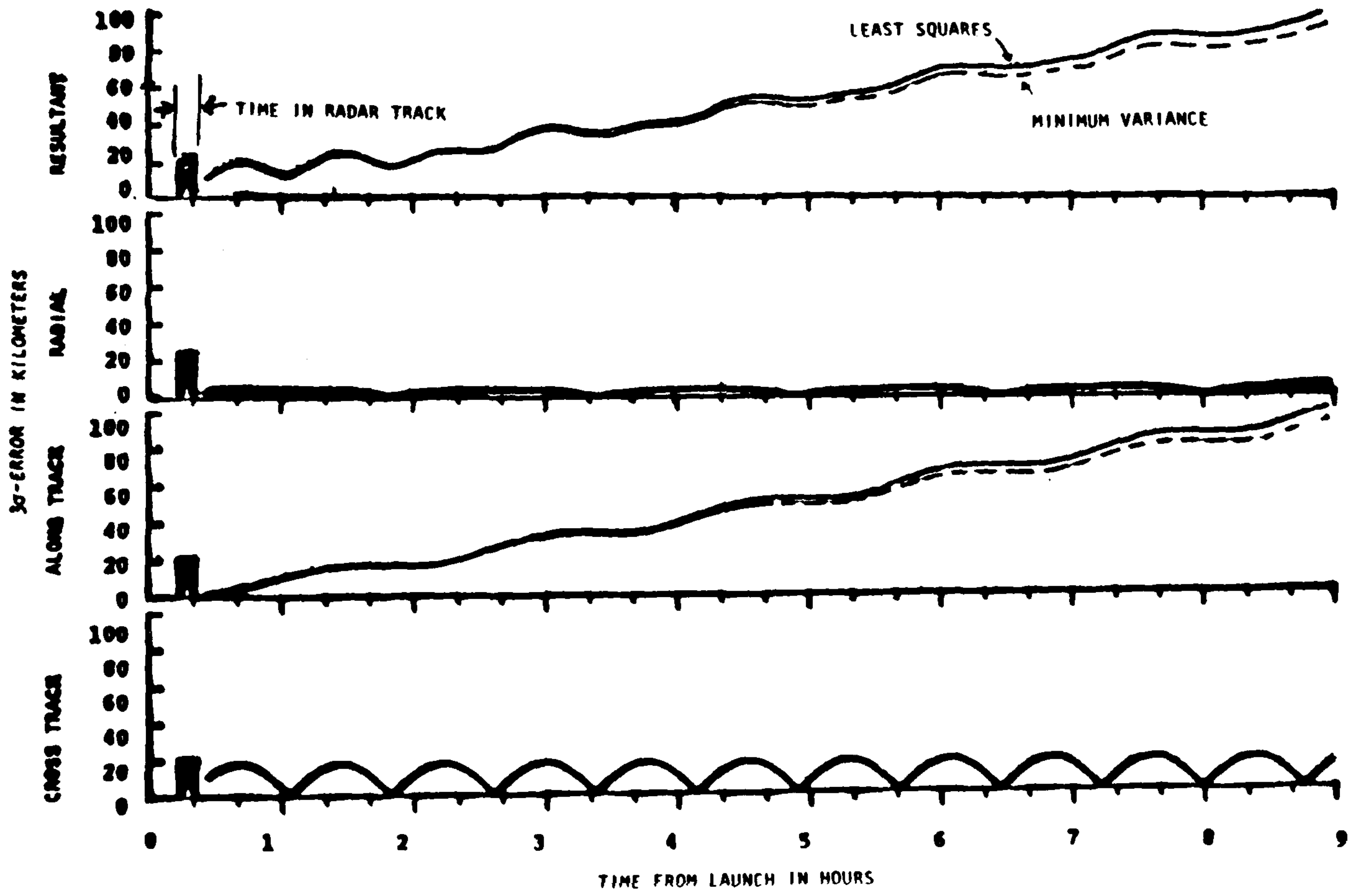


FIGURE 4
EPHERMIS ERRORS:
SINGLE-PASS WITH ZERO RESIDUAL-BIAS ERRORS
(Facsimile of Computer Output)

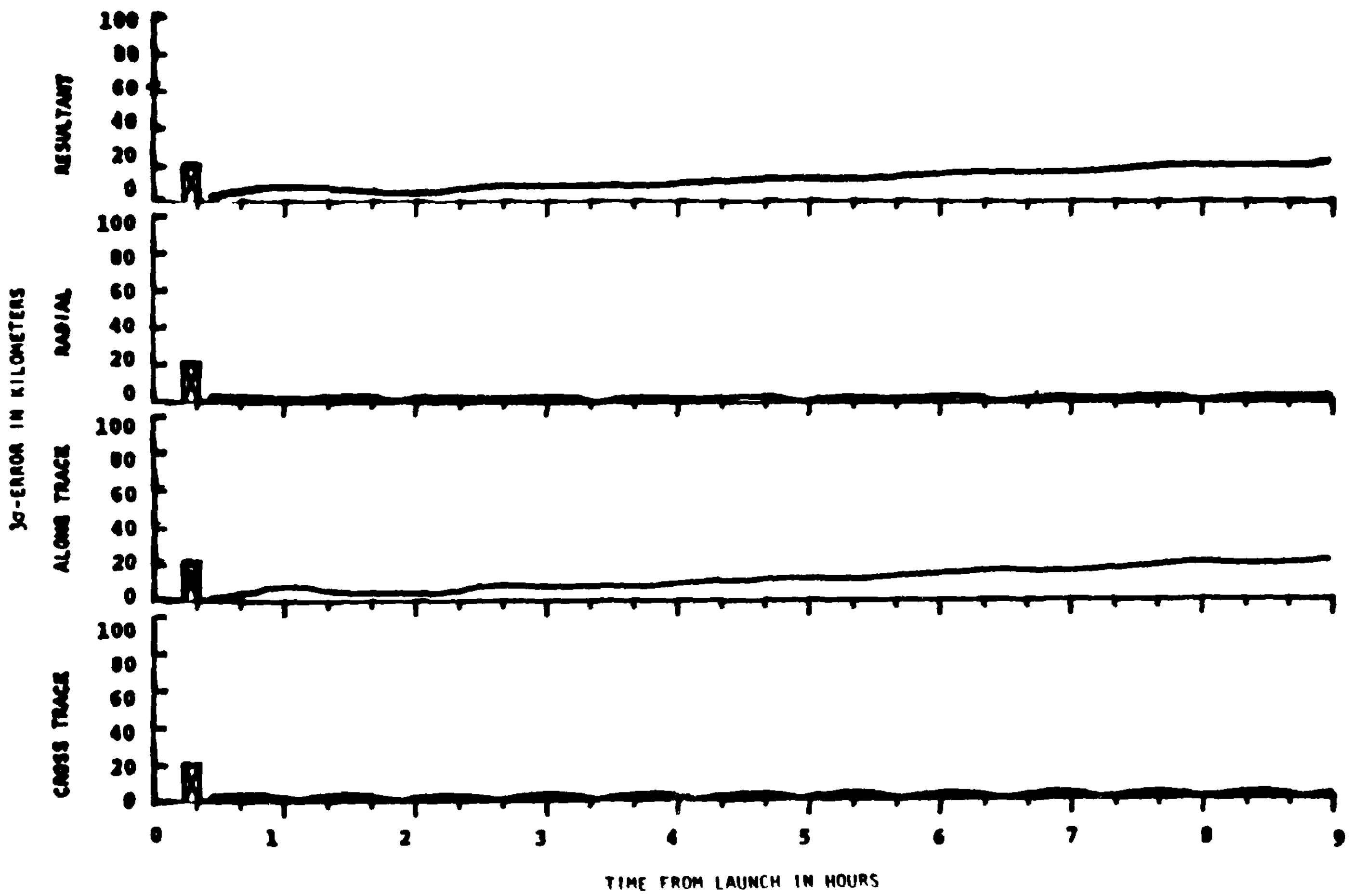


FIGURE 5
EPHERMIS ERROR:
TWO PASSES, QUARTER-REVOLUTION SPACING
(Facsimile of Computer Output)

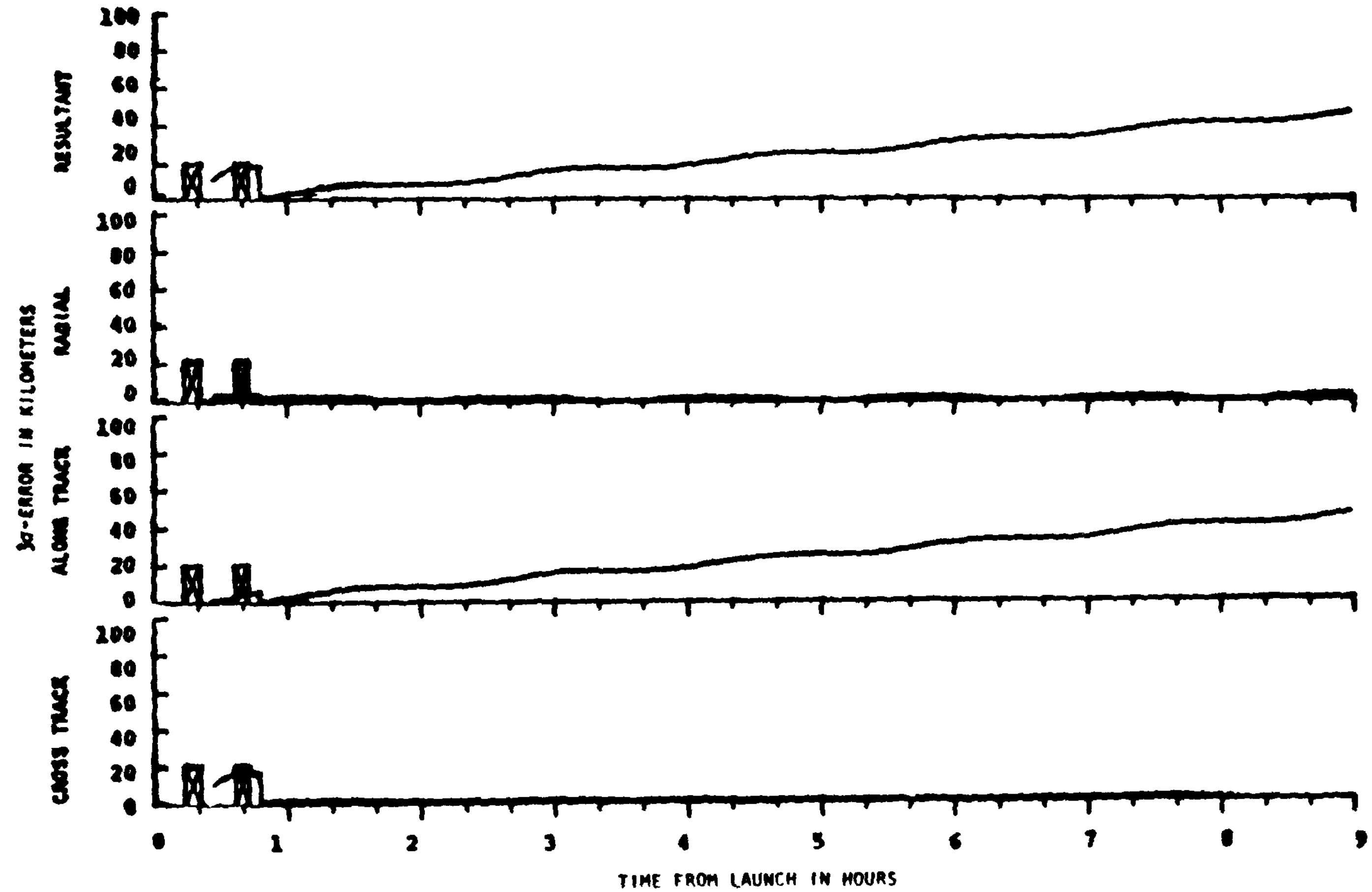
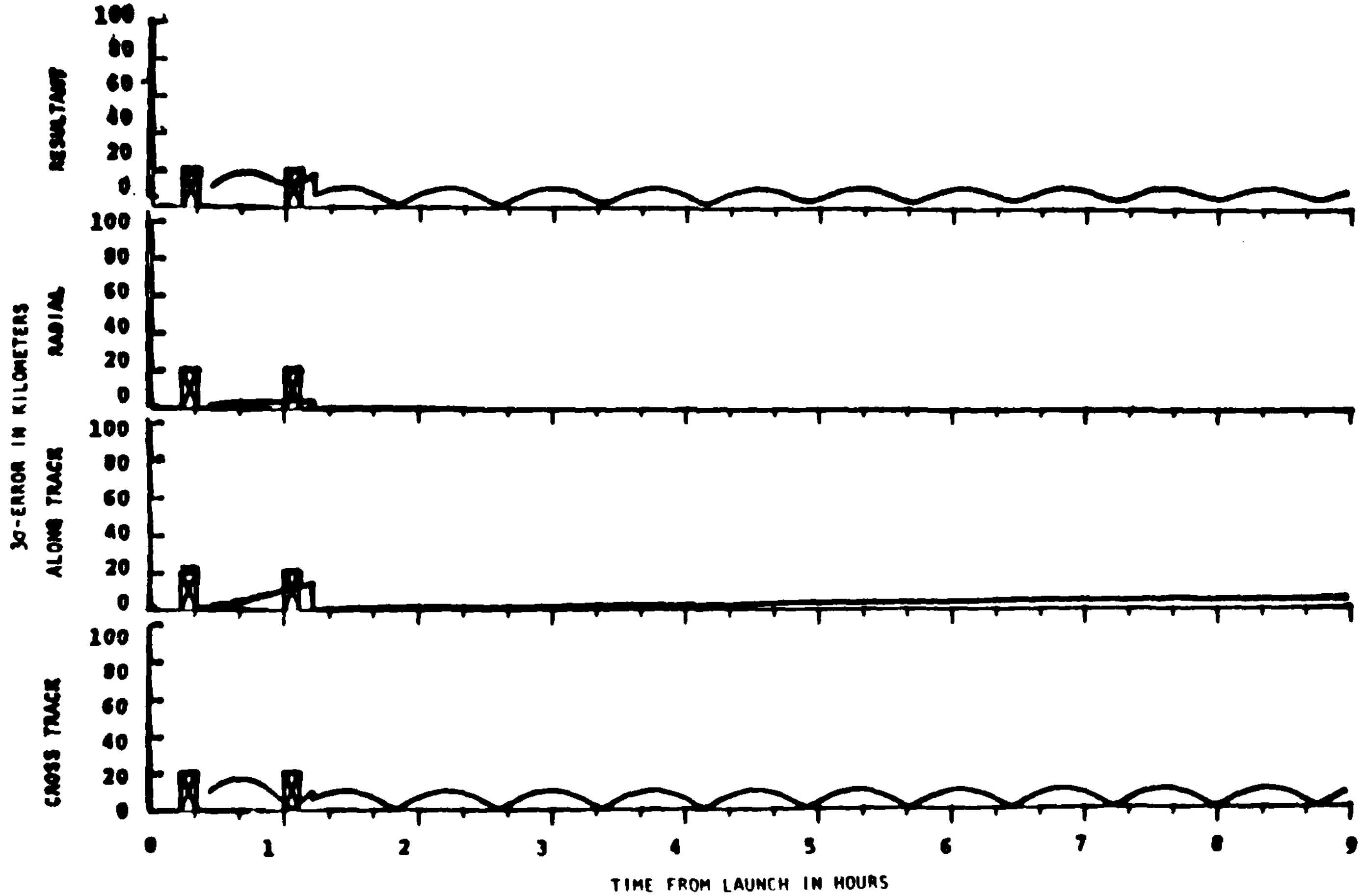


FIGURE 6
EPHERMIS ERROR:
TWO PASSES, HALF-REVOLUTION SPACING
(Facsimile of Computer Output)



Third, the cross-track and radial errors clearly possess a second harmonic in addition to the fundamental. One can understand the structure of the cross-track error by remembering that the estimated and true orbital planes intersect on a line running from the (minimum error) measurement point through the center of the Earth and beyond. One thus expects—and obtains—a second cross-track error minimum where the orbits (nearly) intersect 180° away from the pass point.

The more complex structure of the radial error probably has to do with the interplay of two estimated orbit parameters: the argument of perigee and the orbit eccentricity.

Now compare Figure 4 to Figure 3. "Perfect calibration" of the radar (i.e., zero residual biases) yields a dramatic improvement in all ephemeris error components. One can demonstrate, in fact, that most of the improvement results from elimination of the residual bias in elevation.

Figures 5 and 6 both contain a second radar pass, with measurement errors the same as for Figure 3. The additional data in both cases lead to more accurate ephemerides, as one would expect.

Comparing Figures 5 and 6, it is not surprising to find that the larger pass spacing yields a more accurate satellite period, and hence reduced along-track figure error. The radial error is also slightly better for diametrically opposed points on the orbit.

On the other hand, the quarter-orbit spacing of the passes should stabilize the estimated orbital plane (which must contain the center of the Earth) better than half-orbit spacing. Thus, one may understand the reduced cross-track error of Figure 5 relative to that of Figure 6.

In all of the foregoing examples, the pass geometry is such that the satellite flies almost directly over the radar. Reference 5 contains SEEM output examples for other pass geometries, for lower minimum-elevation angles, and for as many as three passes.

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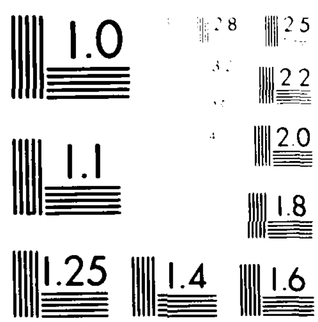

MICROCOPY RESOLUTION TEST CHART

APPENDIX A

COVARIANCE MATRICES: DEFINITION AND PROPERTIES

THE PREDICTION

OF

SATELLITE EPHEMERIS ERRORS

AS THEY RESULT FROM

SURVEILLANCE-SYSTEM MEASUREMENT ERRORS

APPENDIX A

COVARIANCE MATRICES: DEFINITION AND PROPERTIES

This Appendix provides a review of the definition and primary properties of covariance matrices. The first subsection gives properties not restricted to any particular probability density function. The second subsection treats covariance matrices for the special case of normal (Gaussian) distributions.

1. Definition and General Properties

Consider an arbitrary random n-vector

$$w = \mu + \eta \quad , \tag{A-1}$$

where μ is the true value of w and η is some random error, such as measurement error.

Let the probability density function of η be $p(\eta)$. The probability density of the ith component is then

$$p_i(\eta_i) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p(\eta) \prod_{k \neq i} (d\eta_k) \quad . \tag{A-2}$$

(The subscript on p_i indicates that its functional form may depend upon the value of i.) The joint probability density of η_i and η_j (where $i \neq j$) is

$$p_{ij}(\eta_i, \eta_j) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p(\eta) \prod_{\substack{k \neq i \\ k \neq j}} (d\eta_k) \quad . \tag{A-3}$$

Clearly this function is symmetric:

$$p_{ij}(\eta_i,\eta_j) = p_{ji}(\eta_j,\eta_i) \quad . \tag{A-4}$$

With these definitions, the probability that η_i lies in the interval [a,b] is then

$$\int_a^b p_i(\eta_i)d\eta_i \quad .$$

The probability that η_i lies in [a,b], while also η_j lies in [c,d] is

$$\int_c^d \int_a^b p_{ij}(\eta_i,\eta_j)d\eta_i d\eta_j \quad .$$

Now define the *expectation value* of any function $f(\eta)$ as

$$E\{f(\eta)\} = \int_{-\infty}^{+\infty} f(\eta)p(\eta) \prod_{k=1}^{n} (d\eta_k) \quad . \tag{A-5}$$

Then

$$E\{\eta_i\} = \int_{-\infty}^{+\infty} \eta_i p(\eta_i)d\eta_i$$

$$\equiv \bar{\eta}_i \quad , \tag{A-6}$$

the *mean* of η_i. If $\bar{\eta} = 0$, then η is said to be *unbiased*.

Also,

$$E\{(\eta_i-\bar{\eta}_i)^2\} = \int_{-\infty}^{+\infty} (\eta_i-\bar{\eta}_i)^2 p_j(\eta_i)d\eta_i$$

$$\equiv \sigma_i^{\,2} \quad , \tag{A-7}$$

the *variance* of η_i. Further,

$$E\{(\eta_i-\bar{\eta}_i)(\eta_j-\bar{\eta}_j)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\eta_i-\bar{\eta}_i)(\eta_j-\bar{\eta}_j)p_{ij}(\eta_i,\eta_j)d\eta_i d\eta_j$$

$$\equiv \rho_{ij}\sigma_i\sigma_j \quad , \tag{A-8}$$

the *covariance* of η_i and η_j. Define quantity ρ_{ij} as the *correlation coefficient* of η_i and η_j. A theorem exists that

$$-1 \le \rho_{ij} \le +1 \quad . \tag{A-9}$$

Finally, define the *covariance matrix* C_η a matrix with diagonal elements

$$(C_\eta)_{ii} = \sigma_i^{\,2} \quad , \tag{A-10a}$$

and off-diagonal elements

$$(C_\eta)_{ij} = \rho_{ij}\sigma_i\sigma_j \quad . \tag{A-10b}$$

That is,

$$C_\eta = E\{(\eta-\bar{\eta})(\eta-\bar{\eta})^\dagger\} \quad . \tag{A-11}$$

Note that one may estimate C_η from a set of data samples $(\eta)_k$, $k = 1, 2, \ldots, K$ by replacing the integrals of Equations (A-6) and (A-7) by sums over the index k, providing that the joint probability functions $p_{ij}(\eta_i, \eta_j)$ are known (or assumed). (This report does not pursue further the subject of sampling and subsequent estimation of C_η.)

Clearly C_η is symmetric. If all the η_i are linearly independent, then

$$-1 < \rho_{ij} < +1 \tag{A-12}$$

and one can show that C_η is positive definite—i.e., its inverse C_η^{-1} exists.

Now consider the transformation

$$\zeta = F\eta \quad , \tag{A-13}$$

where ζ is an m-vector ($m<n$) and F is any $m \times n$ matrix that is not a function of η.

A functional dependence of ζ on η, such as that in Equation (A-13), means that for every point occurring in η-space a corresponding point exists in ζ-space. Thus, for any arbitrary volume V_η in η-space, the same number of points must occur in the corresponding volume V_ζ in ζ-space:

$$\int_{V_\zeta} \cdots \int p_\zeta(\zeta)\, d\zeta_1 \cdots d\zeta_m = \int_{V_\eta} \cdots \int p_\eta(\eta)\, d\eta_1 \cdots d\eta_n \quad . \tag{A-14}$$

Here the subscripts on the probability densities denote that their functional forms are in general different. But since V_η is arbitrary in Equation (A-14), it follows that

$$p_\zeta(\zeta)\,d\zeta_1\cdots d\zeta_m = p_\eta(\eta)\,d\eta_1\cdots d\eta_n \quad . \qquad \text{(A-15)}$$

This is a fundamental invariance relation for functional transformations among random-vector spaces.

It now follows that

$$E\{\zeta\} \equiv \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \zeta\, p_\zeta(\zeta)\,d\zeta_1\cdots d\zeta_m$$

$$= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} F\eta\cdot p_\eta(\eta)\,d\eta_1\cdots d\eta_n$$

$$= F\cdot E\{\eta\} \quad , \qquad \text{(A-16)}$$

that is,

$$\bar{\zeta} = F\bar{\eta} \quad . \qquad \text{(A-17)}$$

Similarly,

$$C_\zeta = F\cdot C_\eta\cdot F^\dagger \quad . \qquad \text{(A-18)}$$

Note that neither Equation (A-17) nor (A-18) is necessarily valid if $F = F(\eta)$.

According to Equation (A-17), if η is unbiased then ζ will also be unbiased. Equation (A-18) tells how to calculate the covariance matrix in ζ-space. One can show that the symmetry property of C_η is passed on to C_ζ. Furthermore, if C_η^{-1} exists, and if F is of rank m, then C_ζ^{-1} exists and is also symmetric.

Consider the special case in which the transformation matrix is Q, where Q is an n x m matrix of full rank such that

$$QQ^\dagger = I \quad , \tag{A-19a}$$

that is,

$$Q^\dagger = Q^{-1} \quad . \tag{A-19b}$$

Examples of such matrices are those that perform orthogonal rotations upon the space η. Of particular importance is the rotation Q which yields a diagonal C_ζ ("diagonalization of C_η").

Finally, one can show that when F = Q, the determinants of C_ζ and C_η are equal:

$$|C_\zeta| = |C_\eta| \quad . \tag{A-20}$$

Moreover, the trace of C_η is invariant under the Q-transformation:

$$\sum_{i=1}^{n} (\sigma_i{}^2)_\zeta = \sum_{i=1}^{n} (\sigma_i{}^2)_\eta \tag{A-21}$$

Thus, if a variance vector σ is defined with components σ_1, σ_2, ..., σ_n, then Equation (A-20) states that the length of σ is invariant under coordinate-system rotations.

2. Properties When the Density Function Is Normal

Properties of η and C_η treated thus far do not depend upon any particular assumptions about the form of $p(\eta)$. Consider now, however, the following problem.

Referring to Equation (A-1), suppose a measurement of μ is attempted, yielding the measurement w because of noise corruption η. Suppose one knows both $\overline{\eta}$ and C_η. Let the estimated value of μ be

$$\mu^* \equiv w - \overline{\eta} \quad . \tag{A-22}$$

Then the mean of a large number of measurements will be

$$\overline{\mu}^* = \mu \quad . \tag{A-23}$$

The problem is now:

a. What is the probability P that μ^* will fall within some prescribed volume V about the point μ?

b. What is an appropriate prescription for V?

As regards Question a., obviously

$$P = \int_V \cdots \int p(\eta) d\eta_1 \cdots d\eta_n \quad . \tag{A-24}$$

Evaluation of P requires knowledge of the integrand within V. However, $\overline{\eta}$ and C_η, the first and second moments of $p(\eta)$, do not in general completely specify $p(\eta)$—which may possess higher, independent moments. Hence, additional assumptions about $p(\eta)$ are necessary in order to evaluate P.

Moreover, one can expect that the answer to Question b. will be sensitive to the assumed form of $p(\eta)$.

Assume now that $p(\eta)$ has normal (i.e., multivariate Gaussian) form:

$$p(\eta) = \frac{1}{(2\pi)^{n/2}|C_\eta|^{1/2}} e^{-\frac{1}{2}(\eta-\bar{\eta})^\dagger C_\eta^{-1}(\eta-\bar{\eta})} \quad . \tag{A-25}$$

This probability-density form has widespread utility. Since the form is completely specified by $\bar{\eta}$ and C_η, one can expect a mathematically fruitful investigation of the stated problem.

The investigation proceeds by considering the behavior of $p_\eta(\eta)$ under coordinate-system rotations, reverting now to the notation of Equation (A-15). Now under a coordinate rotation

$$\zeta = Q\eta \tag{A-26}$$

the Jacobian of the transformation is the absolute value of the determinant $|Q|$, and so

$$d\zeta_1 \cdots d\zeta_n = ||Q|| d\eta_1 \cdots d\eta_n \quad . \tag{A-27}$$

Then by Equation (A-15)

$$p(\zeta) = \frac{1}{||Q||} p(\eta) \tag{A-28}$$

$$= \frac{1}{(2\pi)^{n/2}||Q|| \cdot |C_\eta|^{1/2}} e^{-\frac{1}{2}(\eta-\bar{\eta})^\dagger C_\eta^{-1}(\eta-\bar{\eta})} \quad . \tag{A-29}$$

Then, using Equation (A-20),

$$||Q|| \cdot |C_\eta|^{1/2} = |C_\zeta|^{1/2}. \tag{A-30}$$

Also, using the rotation-matrix property of Equation (19a),

$$\begin{aligned}(\eta-\bar{\eta})^{\dagger} C_\eta^{-1} (\eta-\bar{\eta}) &= (\eta-\bar{\eta})^{\dagger} Q Q^{\dagger} C_\eta^{-1} Q Q^{\dagger} (\eta-\bar{\eta}) \\ &= (\zeta-\bar{\zeta})^{\dagger} C_\zeta^{-1} (\zeta-\bar{\zeta}) \quad .\end{aligned} \tag{A-31}$$

Hence

$$p_\zeta(\zeta) = \frac{1}{(2\pi)^{n/2} |C_\zeta|^{1/2}} e^{-\frac{1}{2}(\zeta-\bar{\zeta})^{\dagger} C_\zeta^{-1} (\zeta-\bar{\zeta})} \quad , \tag{A-32}$$

which is of the same functional form as Equation (A-29).

One can also readily infer that under coordinate-system translations, the form of Equation (A-29) is also invariant. Let this form be denoted as $p(\bar{\eta}, C_\eta; \eta)$. Thus, the mean and the covariance matrix play the same roles in the probability density function, regardless of coordinate-system selection.

Now assume that Q is chosen such that C_ζ is diagonal. Then

$$p(\bar{\zeta}, C_\zeta; \zeta) = \frac{1}{(2\pi)^{n/2} (\sigma_1 \cdot \sigma_2 \cdots \sigma_n)_\zeta} \cdot$$

$$\cdot e^{-\frac{1}{2}\left[\frac{(\zeta_1-\bar{\zeta}_1)^2}{(\sigma_1{}^2)_\zeta} + \frac{(\zeta_2-\bar{\zeta}_2)^2}{(\sigma_2{}^2)_\zeta} + \cdots + \frac{(\zeta_n-\bar{\zeta}_n)^2}{(\sigma_n{}^2)_\zeta}\right]} \tag{A-33a}$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\ (\sigma_i)_\zeta} e^{-\frac{1}{2} \frac{(\zeta_i - \bar{\zeta}_i)^2}{(\sigma_i^2)_\zeta}} \right] \quad \text{(A-33b)}$$

(i.e., the factor $1/\sqrt{2\Pi}$ occurs n times). Thus, if the covariance matrix is diagonal, the multivariate probability density is the product of univariate densities—i.e., the components of ζ are statistically independent.

This result does not hold in general for non-normal forms of p. However, one can show that statistical independence always implies a diagonal covariance matrix.

This completes the groundwork appropriate to address Question b.

Suppose, in Equation (A-25), the variable expression in the exponent is set equal to a constant a^2, where $a > 0$:

$$(\eta-\bar{\eta})^{\dagger} C_{\eta}^{-1} (\eta-\bar{\eta}) = a^2 \quad . \quad \text{(A-34)}$$

This is the equation of an ellipsoid in the n-fold hyperspace of η. For example, if n = 2, Equation (A-34) becomes

$$\frac{(\eta_1-\bar{\eta})^2}{\sigma_1^2} - 2\rho_{12} \frac{(\eta_1-\bar{\eta}_1)(\eta_2-\bar{\eta}_2)}{\sigma_1\sigma_2} + \frac{(\eta_2-\bar{\eta}_2)^2}{\sigma_2^2} = a^2(1-\rho_{12}^2) \quad . \quad \text{(A-35)}$$

Suppose the integration volume V is taken to be this n-dimensional "error ellipsoid." Then the questions are: What are the properties of this error ellipsoid, and to what extent is it described by C_η^{-1}?

The center of the ellipsoid in η-space is obviously at $\overline{\eta}$. By Equations (A-1) and (A-22),

$$(\mu^* - \mu) = (\eta - \overline{\eta}) \tag{A-36}$$

so that in μ^*-space the ellipsoid center is at μ, which is certainly desirable.

From Equation (A-34) one can see that the surface of the error ellipsoid is a surface of constant probability. For a rotation of the coordinate system to a new ζ-set [see Equation (A-31)], the form of Equation (A-34) is clearly invariant. The relationship of the ellipsoid to the new probability density [Equation (A-32)] is also invariant. Thus, taking Equation (A-20) into account, one can see that the numerical value of the probability density on the ellipsoid surface is invariant under rotations. Since a is arbitrary, it follows that P—given by Equation (A-24)—is also invariant with respect to coordinate-system choice. Thus, the error ellipsoid can meaningfully serve as a "confidence volume."

Assume now that the rotation Q has been selected so as to make C_ζ diagonal. Then, from Equation (A-33a)

$$\frac{(\zeta_1 - \overline{\zeta}_1)^2}{(\sigma_1^2)_\zeta} + \frac{(\zeta_2 - \overline{\zeta}_2)^2}{(\sigma_2^2)_\zeta} + \cdots + \frac{(\zeta_n - \overline{\zeta}_n)^2}{(\sigma_n^2)_\zeta} = a^2 \; . \tag{A-37}$$

The principal half-axes of the ellipsoid are clearly $a(\sigma_1)_\zeta$, $a(\sigma_2)_\zeta$, ..., $a(\sigma_n)_\zeta$. The ellipsoid orientation in η-coordinates is specified by the requisite diagonalization matrix Q. That is, Q is the set of direction cosines between the η-axes and the ζ-axes along which the ellipsoid is oriented.

Thus, C (and of course also C^{-1}) contains information that specifies both the shape and orientation of the error ellipsoid (Q is derivable from C_η). The selectable parameter a is the scaling factor of the principal dimensions of the ellipsoid. Thus

$$V = V(C_\eta, a, n) \quad . \tag{A-38}$$

The "1σ - confidence volume" is then $V(C_\eta, 1, n)$, the "2σ - confidence volume" is $V(C_\eta, 2, n)$, etc.

One can extract a certain amount of information from C_η and C_η^{-1} without diagonalization. The standard deviation of the resultant of η is the trace of C_η:

$$\sigma = \left[\sum_{i=1}^{n} (C_\eta)_{ii}\right]^{1/2} \tag{A-39}$$

[see Equation (A-21)]. The $(C_\eta^{-1})_{ii}$ are the reciprocals of the intercepts of the 1σ - ellipsoid with the coordinate axes of the η-system.

Question a. can now be answered by integrating Equation (A-24) in the most convenient coordinate frame, i.e., the diagonal frame ζ for $\bar{\zeta} = 0$:

$$P = \int \cdots \int_{V(C_\zeta, a, n)} p(0, C_\zeta, \zeta) d\zeta_1 \cdots d\zeta_n$$

$$= \int \cdots \int_{\substack{\text{Sphere of} \\ \text{Radius } a}} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x_i^2} dx_i \quad , \tag{A-40}$$

using Equation (A-33b) with a change of coordinates

$$x_i \equiv \frac{\zeta_i}{(\sigma_i)_\zeta} \quad . \tag{A-41}$$

Thus P for $n = 1$, $n = 2$, $n = 3$, etc., is respectively the probability associated with a linear confidence interval of length $2a\sigma$, a circular confidence area of radius $a\sigma$, a spherical confidence volume of radius $a\sigma$, etc. Table A-1 gives numerical values of P for some commonly occurring values of n and a. This completes the investigation of Question a. for n-vectors.

Suppose, now, that Questions a. and b. are extended in scope to read as follows:

(1) What is the probability P_m that m specified components of μ^* will fall within some prescribed volume V_m about μ_m (i.e., referring to the specified m-dimensional subspace of μ, $m < n$)?

(2) What is an appropriate prescription for V_m?

As before, these questions will be considered only for the case of normal probability density $p(\bar{\eta}, C_\eta; \eta)$.

TABLE A-1
SOME USEFUL P VALUES

ELLIPSOID DIMENSION	ELLIPSOID SCALE-FACTOR a			
n	1	2	3	4
1	.68269	.84270	.91673	.95450
2	.39347	.63212	.77687	.86466
3	.19875	.42759	.60837	.73854
4	.09020	.26424	.44217	.59399
5	.03743	.15085	.30001	.45058
6	.01439	.08030	.19115	.32332
7	.00517	.04016	.11500	.22022
8	.00175	.01899	.06564	.14288
9	.00056	.00853	.03570	.08859
10	.00017	.00366	.01858	.05265

To simplify the ensuing notation, assume for the moment that the first m components of μ^* are those specified. Then the probability density for the m components is

$$p(\eta_m) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p(\overline{\eta}, C_\eta; \eta) d\eta_{m+1} \cdots d\eta_n \quad , \tag{A-42}$$

and

$$P_m = \int \cdots \int_{V_m} p(\eta_m) d\eta_1 \cdots d\eta_m \quad . \tag{A-43}$$

The solution of the subspace problem then hinges upon a fundamental theorem of multivariate normal probability. One can prove that

$$p(\eta_m) = \frac{1}{(2\pi)^{m/2} |C_{\eta m}|} e^{-\frac{1}{2}(\overline{\eta}_m - \eta_m)^{+} C_{\eta m}^{-1} (\eta_m - \overline{\eta}_m)}$$

$$= p(\overline{\eta}_m, C_{\eta m}; \eta) \quad . \tag{A-44}$$

Here the covariance matrix $C_{\eta m}$ is the subset array of C_η corresponding to the m specified components of μ^*.

Since the forms of Equations (A-42) and (A-43) are identical to those of Equations (A-25) and (A-24), respectively, all of the previous results for V and P of the full n-vector immediately follow where one substitutes η_m and $C_{\eta m}$ for V and C_η.

Extension to *any* m components of μ^* is obvious, since the numbering of the components is arbitrary. One must, of course, select corresponding components in forming η_m and $C_{\eta m}$.

Finally, up to this point the analysis has assumed that both C_η and $\overline{\eta}$ are known. If only $\overline{\eta}$ is known, however, one can find μ^* but not P. If only C_η is known, one can find P but not μ^*. These comments also apply to m-fold subspaces of these n-vectors.

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APPENDIX B

PROPAGATION ERROR MATRIX

THE PREDICTION

OF

SATELLITE EPHEMERIS ERRORS

AS THEY RESULT FROM

SURVEILLANCE-SYSTEM MEASUREMENT ERRORS

APPENDIX B

PROPAGATION ERROR MATRIX

The first of the following subsections states the problem of determining a matrix representing the error in the propagation matrix [Equation (B-2)]. The second subsection gives the solution approach and the solution itself. The final subsection contains intermediate steps of the analysis.

1. The Propagation Error Matrix Problem

The objective here is to find the effect of error in the state vector

$$x_o = \begin{bmatrix} r_o \\ \hline \dot{r}_o \end{bmatrix} \qquad \text{(B-1)}$$

upon the propagation matrix

$$\Phi(x_o, \hat{t}, \hat{t}_o) = \left[\begin{array}{ccc|ccc} f & 0 & 0 & g & 0 & 0 \\ 0 & f & 0 & 0 & g & 0 \\ 0 & 0 & f & 0 & 0 & g \\ \hline \dot{f} & 0 & 0 & \dot{g} & 0 & 0 \\ 0 & \dot{f} & 0 & 0 & \dot{g} & 0 \\ 0 & 0 & \dot{f} & 0 & 0 & \dot{g} \end{array}\right] \qquad \text{(B-2)}$$

Here (see Reference B-1)*

$$f = 1 - \frac{a}{r_o}(1 - \cos\hat{E}); \qquad \text{(B-3a)}$$

* Equations (B-3a to 3d) are among Equations (7P18) and (7P19) with the convenient abbreviations $p \equiv c_o$, $q \equiv D_o/\sqrt{a}$. Equations (B-4) to (B-6) are among Equations (7B5) - (7B9). Equation (B-8) is Equation (7P14), and Equation (B-9) is obtained by integrating Equation (4G4) with $\hat{E} \equiv E - E_o$ and n defined by Equation (4G7), identical with Equation (B-7) here.

$$g = \left(\frac{r_o}{na}\right) \sin \hat{E} + \frac{q}{n} (1 - \cos \hat{E}); \tag{B-3b}$$

$$\dot{f} = -n\left(\frac{a^2}{rr_o}\right) \sin \hat{E}; \tag{B-3c}$$

$$\dot{g} = 1 - \frac{a}{r} (1 - \cos \hat{E}), \tag{B-3d}$$

with

$$a = \left(\frac{2}{r_o} - \frac{\dot{r}_o{}^2}{\mu}\right)^{-1}, \tag{B-4}$$

$$p = 1 - \frac{r_o}{a}, \tag{B-5}$$

$$q = \frac{r_o \cdot \dot{r}_o}{\sqrt{\mu a}}, \tag{B-6}$$

$$n = \sqrt{\mu}\, a^{-3/2}. \tag{B-7}$$

$$r = a(1 - p \cos \hat{E} + q \sin \hat{E}), \tag{B-8}$$

and

$$\hat{t} - \hat{t}_o = \frac{1}{n}[\hat{E} - p \sin \hat{E} + q(1 - \cos \hat{E})]. \tag{B-9}$$

For Earth satellites $\sqrt{\mu} \approx 3.78808 \times 10^4\ km^{3/2}/min$.

Specifically, the problem is to obtain an explicit representation for Ψ_{tt_o}, the *propagation error matrix*, as defined by

$$\delta\Phi_{\hat{t}\hat{t}_o} \cdot x_o = \chi_{\hat{t}\hat{t}_o} \cdot \delta x_o . \tag{B-10}$$

2. Solution Summary and Results

Using Equations (B-1) and (B-2), the lefthand side of Equation (B-10) is

$$\begin{bmatrix} r_o \delta f + \dot{r}_o \delta g \\ \hline r_o \delta \dot{f} + \dot{r}_o \delta \dot{g} \end{bmatrix} = \begin{bmatrix} r_o\left(\dfrac{\partial f}{\partial r_o}\,\delta r_o + \dfrac{\partial f}{\partial \dot{r}_o}\,\delta \dot{r}_o\right) + \dot{r}_o\left(\dfrac{\partial g}{\partial r_o}\,\partial r_o + \dfrac{\partial g}{\partial \dot{r}_o}\,\partial \dot{r}_o\right) \\ \hline r_o\left(\dfrac{\partial \dot{f}}{\partial r_o}\,\delta r_o + \dfrac{\partial \dot{f}}{\partial \dot{r}_o}\,\delta \dot{r}_o\right) + \dot{r}_o\left(\dfrac{\partial \dot{g}}{\partial \dot{r}_o}\,\partial r_o + \dfrac{\partial \dot{g}}{\partial \dot{r}_o}\,\partial \dot{r}_o\right) \end{bmatrix}, \tag{B-11}$$

where

$$\frac{\partial f}{\partial r_o} \equiv \begin{bmatrix} \dfrac{\partial f}{\partial x_o} & \dfrac{\partial f}{\partial y_o} & \dfrac{\partial f}{\partial z_o} \end{bmatrix}, \tag{B-12}$$

the quantities x_o, y_o, z_o being the components of r_o; etc. Comparing the Equations (B-11) and (B-10),

$$\Psi_{\hat{t}\hat{t}_o} = \begin{bmatrix} r_o\left[\dfrac{\partial f}{\partial r_o} \;\vdots\; \dfrac{\partial f}{\partial \dot{r}_o}\right] + \dot{r}_o\left[\dfrac{\partial g}{\partial r_o} \;\vdots\; \dfrac{\partial g}{\partial \dot{r}_o}\right] \\ \hline r_o\left[\dfrac{\partial \dot{f}}{\partial r_o} \;\vdots\; \dfrac{\partial \dot{f}}{\partial \dot{r}_o}\right] + \dot{r}_o\left[\dfrac{\partial \dot{g}}{\partial r_o} \;\vdots\; \dfrac{\partial \dot{g}}{\partial \dot{r}_o}\right] \end{bmatrix} . \tag{B-13}$$

The next subsection shows that

$$\frac{\partial f}{\partial r_o} = f_{11} r_o^{\dagger} + f_{12} \dot{r}_o^{\dagger} \quad , \tag{B-14a}$$

$$\frac{\partial f}{\partial \dot{r}_o} = f_{21} r_o^{\dagger} + f_{22} \dot{r}_o^{\dagger} \quad , \tag{B-14b}$$

where

$$f_{11} = \left(\frac{2a^2}{r_o^3}\right)\frac{\partial f}{\partial a} + \left(\frac{1+p}{r_o^2}\right)\frac{\partial f}{\partial p} - \left(\frac{qa}{r_o^3}\right)\frac{\partial f}{\partial q} \quad ; \tag{B-15a}$$

$$f_{12} = \left(\frac{1}{na^2}\right)\frac{\partial f}{\partial q} \quad , \tag{B-15b}$$

$$f_{21} = \left(\frac{1}{na^2}\right)\frac{\partial f}{\partial q} \quad , \tag{B-15c}$$

$$f_{22} = \left(\frac{2}{n^2 a}\right)\frac{\partial f}{\partial a} + \left(\frac{2r_o}{n^2 a^3}\right)\frac{\partial f}{\partial p} - \left(\frac{q}{n^2 a^2}\right)\frac{\partial f}{\partial q} \quad . \tag{B-15d}$$

Identical forms exist in g, $\dot{f}$, and $\dot{g}$. Using Equations (B-14) and their analogs, the required result is

$$\Psi_{\hat{t}\hat{t}_o} = \left[\begin{array}{c|c} f_{11} r_o \cdot r_o^{\dagger} + f_{12} r_o \cdot \dot{r}_o^{\dagger} + & f_{21} r_o \cdot r_o^{\dagger} + f_{22} r_o \cdot \dot{r}_o^{\dagger} + \\ + g_{11} \dot{r}_o \cdot r_o^{\dagger} + g_{12} \dot{r}_o \cdot \dot{r}_o^{\dagger} & + g_{21} \dot{r}_o \cdot r_o^{\dagger} + g_{22} \dot{r}_o \cdot \dot{r}_o^{\dagger} \\ \hline \dot{f}_{11} r_o \cdot r_o^{\dagger} + \dot{f}_{12} r_o \cdot \dot{r}_o^{\dagger} + & \dot{f}_{21} r_o \cdot r_o^{\dagger} + \dot{f}_{22} r_o \cdot \dot{r}_o^{\dagger} + \\ + \dot{g}_{11} \dot{r}_o \cdot r_o^{\dagger} + \dot{g}_{12} \dot{r}_o \cdot \dot{r}_o^{\dagger} & + \dot{g}_{21} \dot{r}_o \cdot r_o^{\dagger} \quad \dot{g}_{22} \dot{r}_o \cdot \dot{r}_o^{\dagger} \end{array}\right] \tag{B-16}$$

The partial derivatives of Equations (B-15) and of its analogs in g, f, and g are given in Table B-1, wherein:

$$\hat{E}_a \equiv \left(\frac{a}{r} \cdot \frac{3}{2a}\right) n(\hat{t} - \hat{t}_o) \qquad \text{(B-17a)}$$

$$\hat{E}_p \equiv \frac{a}{r} \sin \hat{E} \quad , \qquad \text{(B-17b)}$$

$$\hat{E}_q \equiv - \frac{a}{r}(1-\cos \hat{E}) ; \qquad \text{(B-17c)}$$

$$w \equiv p \sin \hat{E} + q \cos \hat{E} \quad . \qquad \text{(B-18)}$$

Note that expressions for the derivatives of f, g, $\dot{f}$, and $\dot{g}$, with respect to each of the components of r_O and r_O, are available in Reference B-2. There, development of derivatives proceeds from expressions of the general form of Equations (B-14) and (B-15) here, except that the intermediate variables D_O, r_O, and 1/a are used rather than p, q, and a. One can prove that the results here are identical to those of Reference B-2, after correcting certain misprints in the latter.*

3. <u>Derivation of Partial Derivatives</u>

This subsection contains derivations of Equations (B-14), (B-15), and results of Table B-1 with defining Equations (B-17) and (B-18).

---

* In Reference B-2, Equation (15C20), the term $\frac{\partial \hat{M}}{\partial \alpha}$ has been omitted [see Equation (15C5)]. In Equation (15C53) for $\dot{g}_r$, the first denominator should read $r_O r^3$ rather than $r_O{}^3$.

TABLE B-1
PARTIAL DERIVATIVES

$\frac{\partial f}{\partial a} = - \hat{E}_a \left(\frac{a}{r_o}\right) \sin \hat{E}$	$\frac{\partial g}{\partial a} = \frac{1}{n} \Big\} \left(\frac{3}{2a}\right) \Big[\left(\frac{r_o}{a}\right) \sin \hat{E}$ $+ q(1-\cos \hat{E}) \Big]$ $+ \Big[\left(\frac{r_o}{a}\right) \cos \hat{E}$ $+ q \sin \hat{E} \Big] \cdot \hat{E}_a \Big\}$	$\frac{\partial \dot{f}}{\partial a} = n \left(\frac{a^2}{r r_o}\right) \Big\{ \left(\frac{3}{2a}\right) \sin \hat{E}$ $- \hat{E}_a \Big[\cos \hat{E}$ $- w \left(\frac{a}{r}\right) \sin \hat{E} \Big] \Big\}$	$\frac{\partial \dot{g}}{\partial a} = -\hat{E}_a \left(\frac{a}{r}\right) \Big[\sin \hat{E}$ $-w \left(\frac{a}{r}\right) (1-\cos \hat{E}) \Big]$
$\frac{\partial f}{\partial p} = -(1-\cos \hat{E}) \left(\frac{a}{r_o}\right)^2$ $- E_p \left(\frac{a}{r_o}\right) \sin \hat{E}$	$\frac{\partial g}{\partial p} = \frac{1}{n} \Big\{ -\sin \hat{E}$ $+ E_p \Big[\left(\frac{r_o}{a}\right) \cos \hat{E}$ $+ q \sin \hat{E} \Big] \Big\}$	$\frac{\partial \dot{f}}{\partial p} = -n \left(\frac{a^2}{r r_o}\right) \Big\{ \sin \hat{E} \Big[\frac{a}{r_o}$ $+ \left(\frac{a}{r}\right) \cos \hat{E} \Big] + \hat{E}_p \Big[\cos \hat{E}$ $- w \left(\frac{a}{r}\right) \sin \hat{E} \Big] \Big\}$	$\frac{\partial \dot{g}}{\partial p} = - \left(\frac{a}{r}\right) \Big\{ \left(\frac{a}{r}\right) \cos \hat{E} (1-\cos \hat{E})$ $+ \hat{E}_p \Big[\sin \hat{E}$ $- w \left(\frac{a}{r}\right) (1-\cos \hat{E}) \Big] \Big\}$
$\frac{\partial f}{\partial q} = - \hat{E}_q \left(\frac{a}{r_o}\right) \sin \hat{E}$	$\frac{\partial g}{\partial q} = - \frac{1}{n} \Big\{ (1-\cos \hat{E})$ $+ \hat{E}_q \Big[\left(\frac{r_o}{a}\right) \cos \hat{E}$ $+ q \sin \hat{E} \Big] \Big\}$	$\frac{\partial \dot{f}}{\partial q} = n \left(\frac{a^2}{r r_o}\right) \Big\{ \left(\frac{a}{r}\right) \sin^2 \hat{E}$ $- \hat{E}_q \Big[\cos \hat{E}$ $- w \left(\frac{a}{r}\right) \sin \hat{E} \Big] \Big\}$	$\frac{\partial \dot{g}}{\partial q} = \left(\frac{a}{r}\right) \Big\{ \left(\frac{a}{r}\right) \sin \hat{E} (1-\cos \hat{E})$ $+ E_q \Big[-\sin \hat{E}$ $+ w \left(\frac{a}{r}\right) (1-\cos \hat{E}) \Big] \Big\}$

Referring to the first footnote of this appendix, Reference B-1 shows that

$$p = e \cos E_o \quad , \tag{B-19a}$$

$$q = e \sin E_o \quad . \tag{B-19b}$$

Here e is the orbit eccentricity and E_o is the eccentric anomaly at epoch $\hat{t}_o$. One will find that of the parameters a, p, q, n, and r, only three can be independently chosen.

Let these three be a, p, and q. Then if ξ is any position or velocity component,

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial \xi} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial \xi} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial \xi} \tag{B-20}$$

with analogous relations in g, $\dot{f}$, and $\dot{g}$. From Equations (B-4) through (B-6) one can determine that

$$\frac{\partial a}{\partial x_o} = \left(\frac{2a^2}{r_o^3}\right) \cdot x_o \quad , \tag{B-21a}$$

$$\frac{\partial p}{\partial x_o} = \frac{1}{r_o^2}(1+p)\,x_o \quad , \tag{B-21b}$$

$$\frac{\partial q}{\partial x_o} = -\left(\frac{qa}{r_o^3}\right)x_o + \left(\frac{1}{na^2}\right)\dot{x}_o \quad , \tag{B-21c}$$

with analogs in y_o and z_o. Similarly

$$\frac{\partial a}{\partial \dot{x}_o} = \left(\frac{2}{n^2 a}\right) \cdot \dot{x}_o \quad , \tag{B-22a}$$

$$\frac{\partial p}{\partial \dot{x}_o} = \left(\frac{2r_o}{n^2 a^3}\right) \cdot \dot{x}_o \quad , \tag{B-22b}$$

$$\frac{\partial q}{\partial \dot{x}_o} = \left(\frac{1}{na^2}\right) x_o - \left(\frac{q}{n^2 a^2}\right) \dot{x}_o \quad , \tag{B-22c}$$

with analogs in $\dot{y}_o$ and $\dot{z}_o$. Equations (B-14) and (B-15), and analogs in g, $\dot{f}$, and $\dot{g}$, follow immediately by combining Equations (B-20), (B-21), and (B-22).

Next, note that

$$\frac{\partial n}{\partial a} = - \frac{3n}{2a} \quad , \tag{B-23a}$$

$$\frac{\partial n}{\partial p} = 0 \quad , \tag{B-23b}$$

$$\frac{\partial n}{\partial q} = 0 \quad . \tag{B-23c}$$

Also, for an arbitrary variable η,

$$\begin{aligned} \frac{\partial}{\partial \eta}\left(\frac{a}{r_o}\right) &= -\left(\frac{a}{r_o}\right)^2 \frac{\partial}{\partial \eta}\left(\frac{r_o}{a}\right) \\ &= \left(\frac{a}{r_o}\right)^2 \frac{\partial p}{\partial \eta} \end{aligned} \tag{B-24}$$

by Equation (B-5). Hence

$$\frac{\partial}{\partial a}\left(\frac{a}{r_o}\right) = 0 \quad , \tag{B-25a}$$

$$\frac{\partial}{\partial p}\left(\frac{a}{r_o}\right) = \left(\frac{a}{r_o}\right)^2 \quad , \tag{B-25b}$$

$$\frac{\partial}{\partial q}\left(\frac{a}{r_o}\right) = 0 \quad . \tag{B-25c}$$

From Equations (B-8) and (B-18),

$$\frac{\partial}{\partial a}\left(\frac{a}{r}\right) = -\left(\frac{a}{r}\right)^2 w \frac{\partial \hat{E}}{\partial a} \quad , \tag{B-26a}$$

$$\frac{\partial}{\partial p}\left(\frac{a}{r}\right) = -\left(\frac{a}{r}\right)^2 \left(-\cos \hat{E} + w \frac{\partial \hat{E}}{\partial p}\right) , \tag{B-26b}$$

$$\frac{\partial}{\partial q} \frac{a}{r} = -\left(\frac{a}{r}\right)^2 \left(\sin \hat{E} + w \frac{\partial \hat{E}}{\partial p}\right) \quad . \tag{B-26c}$$

Differentiation of Equations (B-9) yields Equations (B-17), where

$$\frac{\partial \hat{E}}{\partial a} \equiv \hat{E}_a \quad , \tag{B-27a}$$

$$\frac{\partial \hat{E}}{\partial p} \equiv \hat{E}_p \quad , \tag{B-27b}$$

$$\frac{\partial \hat{E}}{\partial q} \equiv \hat{E}_q \quad . \tag{B-27c}$$

With the aid of Equations (B-23), (B-25), and (B-26) one can then differentiate Equations (B-3) with respect to a, p, and q to obtain the results in Table B-1.

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B-2. Herrick, S., *Astrodynamics, Vol. 2*, Van Nostrand Reinhold, London, 1972, pp. 51-55.

APPENDIX C

INVERSION OF A MATRIX

THE PREDICTION
OF
SATELLITE EPHEMERIS ERRORS
AS THEY RESULT FROM
SURVEILLANCE-SYSTEM MEASUREMENT ERRORS

APPENDIX C

INVERSION OF A MATRIX

Suppose a matrix has the form

$$
C = \begin{bmatrix}
G_1+H & \cdots & H & H & H & H \\
\vdots & & \vdots & \vdots & \vdots & \vdots \\
H & \cdots & G_{N-3}+H & H & H & H \\
H & \cdots & H & G_{N-2}+H & H & H \\
H & \cdots & H & H & G_{N-1}+H & H \\
H & \cdots & H & H & H & G_N+H
\end{bmatrix} , \tag{C-1}
$$

wherein the G_I, $I=1,2,\ldots,N$, and H are themselves $n \times n$ diagonal matrices with elements, for $i=1,2,\ldots,n$:

$$
(G_I)_{ii} > 0 \quad , \tag{C-2}
$$

$$
H_{ii} \geq 0 \quad . \tag{C-3}
$$

The problem here is to find C^{-1}, and concomitantly to demonstrate that Equations (C-2) and (C-3) ensure its existence.

The ensuing solution rests upon a generalization of Cramer's Rule (Reference C-1, p. 286). Given a matrix whose partitions are members of a field, the determinant of the matrix is also a member of the field and can be evaluated by standard manipulative techniques. Cramer's Rule then holds for finding the inverse matrix, whose corresponding partitions will again be members of the field.

Now, the set of n x n invertible, diagonal matrices plug the null matrix, forms a field under matrix addition and multiplication. Both G_I and H are members of this field, assuming for the moment that H as well as G_I is invertible. Thus

$$|C| \cdot (C^{-1})_{IJ} = |\Delta|_{IJ} \; ; \; I,J = 1,2,\cdots,N \quad , \tag{C-4}$$

an n x n matrix equation wherein all quantities are to be expressed in terms of G_I and H. Here $|C|$ is the determinant of C. The quantity $|\Delta|_{IJ}$ is the cofactor of C_{JI}, and $(C^{-1})_{IJ}$ is the general element of C^{-1}.

One may use a recursive approach to evaluate the determinant $|C|$. Let

$$C_N \equiv C \quad . \tag{C-5}$$

Then

$$C_1 = G_1 + H \quad ,$$

and for N>1 let C_{N-1} be the matrix of the first N-1 rows and columns of C_N.

Now one may expand $|C_N|$ via the last row, and after some rearrangement of rows and columns obtain the following:

$$
\begin{aligned}
|C_N| &= (G_N + H)\cdot|C_{N-1}| \\
&- H \begin{vmatrix} G_1+H & \cdots & H & H & H \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ H & \cdots & G_{N-3}+H & H & H \\ H & \cdots & H & G_{N-2}+H & H \\ H & \cdots & H & H & H \end{vmatrix} \\
&- H \begin{vmatrix} G_1+H & \cdots & H & H & H \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ H & \cdots & G_{N-3}+H & H & H \\ H & \cdots & H & G_{N-1}+H & H \\ H & \cdots & H & H & H \end{vmatrix} \\
&- \cdots - H \begin{vmatrix} G_2+H & \cdots & H & H & H \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ H & \cdots & G_{N-3}+H & H & H \\ H & \cdots & H & G_{N-2}+H & H \\ H & \cdots & H & H & H \end{vmatrix} .
\end{aligned}
\tag{C-6}
$$

One may evaluate the determinants here by subtracting the last row from each of the other rows and then expanding via the last column. Equation (C-6) then becomes

$$|C_N| = (G_N + B)\,|C_{N-1}| + H^2\left(\prod_{I=1}^{N-1} G_I\right)\sum_{I=1}^{N-1} G_I^{-1} \ . \tag{C-7}$$

By evaluating $|C_2|$ and then $|C_3|$, one may infer that $|C_N|$ probably has the form

$$|C_N| = \left(\prod_{I=1}^{N} G_I\right)\left(1 + H\sum_{I=1}^{N} G_I^{-1}\right) \ . \tag{C-8}$$

One may easily prove this result, for $N \geq 1$, by substitution into Equation (C-7).

Clearly $|C_N|$ is $n \times n$ diagonal, and

$$|C_N|_{ii} = \left[\prod_{I=1}^{N} (G_I)_{ii}\right]\cdot\left[1 + H_{ii}\sum_{I=1}^{N}\left(1/(G_I)_{ii}\right)\right] \ . \tag{C-9}$$

By Equations (C-2) and (C-3), none of the $|C_N|_{ii}$ vanishes, so that $|C_N|^{-1}$ exists. Hence by Equation (C-4) all of the C_{IJ}^{-1} exist and so C^{-1} exists.

Now in general $|\Delta_{IJ}|$ is an $(N-1) \times (N-1)$ matrix. Define

$$|\Delta_{IJ}|_{N-1} = |\Delta_{IJ}| \quad , \tag{C-10}$$

where for consistency of the preceding results

$$|\Delta_{IJ}|_{o} = 1 \quad , \tag{C-11}$$

the n x n identity matrix. The values of $|\Delta_{IJ}|_1$, corresponding to N=2, are clearly

$$|\Delta_{11}|_1 = G_2 + H \ ; \ |\Delta_{12}|_1 = -H \ ;$$

$$|\Delta_{21}|_1 = -H \ ; \ |\Delta_{22}|_1 = G_1 + H \quad .$$

For N>2, one can find $|\Delta_{IJ}|_{N-1}$ by again using a recursion approach. For the cases I=J, one may find $|\Delta_{II}|_{N-1}$ via exactly the approach used previously for $|C_N|$:

$$|\Delta_{II}|_{N-1} = G_I^{-1}\left(\prod_{K=1}^{N} G_K\right)\left(1 + H\sum_{K=1}^{N} G_K^{-1} - H\,G_I^{-1}\right) . \tag{C-12}$$

Observe that this result holds for $N \geq 1$.

For $I \neq J$, a typical $|\Delta_{IJ}|_{N-1}$ for large N is

$$|\Delta_{N-2,N-3}|_{N-1} = (-1)^{N-2}(-1)^{N-3} \begin{vmatrix} G_1+H & \cdots & H & H & H \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ H & \cdots & G_{N-3}+H & H & H \\ H & \cdots & H & H & H \\ H & \cdots & H & H & G_N+H \end{vmatrix} \tag{C-13}$$

By interchanging the last two rows and then the last two columns, one can bring the determinant here into the form of the determinants of Equation (C-6) and use the same evaluation procedure as before. One can then see that

$$\left(|\Delta_{IJ}|_{N-1}\right)_{I\neq J} = -H\ G_I^{-1}\ G_J^{-1}\left(\prod_{K=1}^{N} G_K\right)\ , \qquad \text{(C-14)}$$

valid for $N \geq 2$.

One can then easily verify that

$$C_{IJ}^{-1} = G_I^{-1}\ \delta_{IJ} - H\left(1 + H\sum_{K=1}^{N} G_K^{-1}\right)^{-1} G_I^{-1}\ G_J^{-1}\ , \qquad \text{(C-15)}$$

where σ_{IJ} is the Kronecker delta. This holds for $N \geq 1$, $I=1,\ldots,N$ and $J=1,\ldots,N$. This equation also holds even when H is not invertible, since that particular property of H—assumed earlier—was not actually used in the derivation. Finally, one can decompose this diagonal-matrix equation into a set of scalar equations by inspection, with the aid of the relation

$$\left(1 + H\sum_{K=1}^{N} G_K^{-1}\right)_{ii}^{-1} = \frac{1}{1 + H_{ii}\sum_{K=1}^{N}\left(1/\left(G_K\right)_{ii}\right)};\ i = 1,\cdots,n\ . \qquad \text{(C-16)}$$

REFERENCES

C-1. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, MacMillan, N.Y., 1965.

APPENDIX D

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THE PREDICTION
OF
SATELLITE EPHEMERIS ERRORS
AS THEY RESULT FROM
SURVEILLANCE-SYSTEM MEASUREMENT ERRORS

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report derives equations predicting satellite ephemeris error as a function of measurement errors of space-surveillance sensors. These equations lend themselves to rapid computation with modest computer resources. They are applicable over prediction times such that measurement errors, rather than *uncertainties* of atmospheric drag and of Earth shape, dominate in producing ephemeris error. This report describes the specialization of these equations underlying the ANSER computer program, SEEM (Satellite Ephemeris

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20. (Continued) Error Model). The intent is that this report be of utility to users of SEEM for interpretive purposes, and to computer programmers who may need a mathematical point of departure for limited generalization of SEEM.